Recent Developments in Bootstrapping High Frequency Financial Data

Sílvia Gonçalves
University of Montreal

SoFie Annual Conference 2014

Introduction

- High frequency data is widely used in financial econometrics.
- Examples include:
  - estimation of volatility: realized volatility and its extensions.
  - estimation of (functionals of) multivariate volatility:
    - realized covariance matrix
    - realized regression coefficients
    - realized correlations
  - estimation and detection of jumps.
- **Main goal of this talk**: how to bootstrap high frequency data.

Motivation for the bootstrap

- Inference is usually based on infill asymptotics.
- Finite sample distortions:
  - especially true for low to moderate frequencies.
  - but also for higher frequencies due to slower rate of convergence.

Motivation for the bootstrap

- Inference is usually based on infill asymptotics.
- Finite sample distortions:
  - especially true for low to moderate frequencies.
  - but also for higher frequencies due to slower rate of convergence.
- One potential solution: rely on nonlinear transformations, e.g. use the log of RV instead of RV.
  - not universally available.
- Another solution: the bootstrap.

Finite sample properties of pre-averaging approach

Coverage rates of nominal 95% intervals

		SV1F		SV2F	
Frequency	n	$\xi^2 = 0.0001$	$\xi^2 = 0.01$	$\xi^2 = 0.0001$	$\xi^2 = 0.01$
"2-min"	195	90.89	83.11	88.60	80.96
"15-sec"	1560	93.86	87.97	92.62	88.02
"1-sec"	23400	94.80	92.87	94.68	92.88

Based on Jacod et al. (2009) with $\theta = 1/3$ and $k_n = \theta\sqrt{n}$;1 day = 6.5 hours
$Y_t = X_t + \epsilon_t$, $\epsilon_t \sim$ i.i.d.$N\left(0, \omega^2\right)$; $\xi^2 = \omega^2/\sqrt{IQ}$ is the noise-to-signal ratio;

Finite sample properties of tests of H_0 : "no jumps"

Null rejection rates for 5% nominal level test

	SV1F		SV2F	
n	linear	log	linear	log
48	10.24	7.18	15.21	12.54
96	8.46	6.57	12.48	10.82
288	6.87	5.64	9.36	8.62
576	5.99	5.10	7.80	7.25
1152	5.95	5.47	5.78	6.65

Based on Barndorff-Nielsen and Shephard (2004, 2006);
No market microstructure noise.

In this talk: two main questions

1. How to bootstrap high frequency returns contaminated by market microstructure noise?
   - Focus: inference for integrated volatility using pre-averaging.
   - Based on joint work Ulrich Hounyo and Nour Meddahi.
2. How to bootstrap jump tests?
   - Focus: Barndorff-Nielsen and Shephard (2004, 2006) tests based on realized volatility and bipower variation.
   - Based on joint work with Prosper Dovonon, Ulrich Hounyo and Nour Meddahi.

The basic setup: no noise, no jumps

- Stochastic volatility model:

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s.$$

- Intraday $\frac{1}{n}$-returns:

$$r_i = X_{i/n} - X_{(i-1)/n}, \text{ for } i = 1, \ldots, n.$$

- Integrated volatility over a day:

$$IV = \int_0^1 \sigma_s^2 ds.$$

- Realized volatility:

$$RV_n = \sum_{i=1}^{n} r_i^2.$$

Gonçalves and Meddahi (2009): bootstrap for RV_n

- i.i.d. bootstrap:

$$r_i^* \sim \text{i.i.d.} \{r_i : i = 1, \ldots, n\}.$$

  - motivated by a benchmark model where volatility is constant.

- Wild bootstrap:

$$r_i^* = r_i \cdot \eta_i, \quad \text{where} \quad \eta_i \sim \text{i.i.d.}, \ \{\eta_i\} \perp \{r_i\}$$

  - motivated by a toy model with no drift and no leverage:

$$\begin{aligned} r_i|\sigma &\sim N(0, v_i^n), \quad \text{independently across } i, \\ \text{with } v_i^n &= \int_{(i-1)/n}^{i/n} \sigma_s^2 ds. \end{aligned}$$

- Both methods are asymptotically valid under general conditions, including leverage.

Extension of bootstrap results for basic model

- Dovonon, Gonçalves and Meddahi (2011): extension to multivariate context.
  - pairs bootstrap for realized covariance, realized correlation and realized regression.

Extension of bootstrap results for basic model

- Dovonon, Gonçalves and Meddahi (2011): extension to multivariate context.
  - pairs bootstrap for realized covariance, realized correlation and realized regression.
- Hounyo (2013): "local Gaussian bootstrap"

$$\begin{aligned} r_i^* &= \sqrt{\hat{v}_i^n} \cdot \eta_i, \\ \eta_i &\sim \text{i.i.d. } N(0,1) \end{aligned}$$

  - $\hat{v}_i$ is a local RV estimate computed over non-overlapping blocks of r_i.
- The local Gaussian bootstrap provides third-order refinements.
  - related to local Gaussianity approach of Mykland and Zhang (2009).

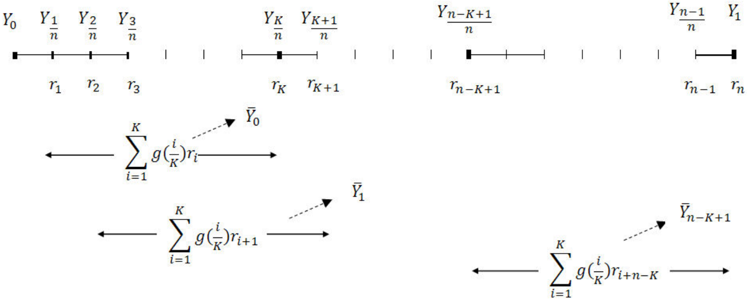
What if there is market microstructure noise?

- We observe
$$Y_t = X_t + \epsilon_t,$$
where ϵ_t represents the noise at time $t = i/n$, for $i = 0, \ldots, n$.
- Then
$$\begin{aligned} r_i &= Y_{i/n} - Y_{(i-1)/n} \\ &= \left(X_{i/n} - X_{(i-1)/n}\right) + \left(\epsilon_{i/n} - \epsilon_{(i-1)/n}\right) \\ &\equiv \underbrace{r_i^e}_{=O_P\left(\frac{1}{\sqrt{n}}\right)} + \underbrace{\Delta\epsilon_i}_{=O_P(1)}. \end{aligned}$$
- Realized volatility is no longer consistent for IV (Zhang et al. (2005) and Bandi and Russell (2008)).

Market microstructure noise robust estimators

- Subsampling (Zhang, Mykland, Aït-Sahalia, 2005).
- Realized kernels (Barndorff-Nielsen, Hansen, Lunde and Shephard, 2008).
- Pre-averaging (Podolskij and Vetter, 2009, and Jacod, Li, Mykland, Podolskij and Vetter, 2009).
- QMLE (Xiu, 2010).

The pre-averaging approach of Jacod et al. (2009)



Asymptotic theory

- Pre-averaged realized volatility:

$$PRV_n = \underbrace{\frac{1}{\psi_2 k_n} \sum_{i=0}^{n-k_n+1} \bar{Y}_i^2}_{\substack{= \widetilde{PRV}_n \\ \downarrow \\ \text{“}RV_n\text{-like”}}} - \underbrace{\frac{\psi_1}{2k_n^2 \psi_2} \sum_{i=1}^{n} r_i^2}_{= bias},$$

  - ψ_1 and ψ_2 are known constants.

Asymptotic theory

- Pre-averaged realized volatility:

$$PRV_n = \underbrace{\frac{1}{\psi_2 k_n} \sum_{i=0}^{n-k_n+1} \bar{Y}_i^2}_{= \widetilde{PRV}_n} - \underbrace{\frac{\psi_1}{2k_n^2 \psi_2} \sum_{i=1}^{n} r_i^2}_{= bias},$$

$\downarrow$

"RV_n-like"

  - ψ_1 and ψ_2 are known constants.

- Jacod et al. (2009): if $k_n = \theta\sqrt{n}$, then as $n \to \infty$,

$$n^{1/4} \left(PRV_n - IV \right) \to^{st} N(0, V).$$

  - the bias term does not influence V.

Which bootstrap should we use?

- The presence of noise
$$r_i = r_i^e + \Delta\epsilon_i$$
generates serial correlation in observed returns.
- Thus, the wild bootstrap is not valid when applied to r_i (nor is the i.i.d. bootstrap).

Which bootstrap should we use?

- The presence of noise
$$r_i = r_i^e + \Delta\epsilon_i$$
generates serial correlation in observed returns.
- Thus, the wild bootstrap is not valid when applied to r_i (nor is the i.i.d. bootstrap).
- Our proposal is to resample the pre-averaged returns:

$$\bar{Y}_i = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right) r_{i+j}.$$

Bootstrapping $\bar{Y}_i$

- We can distinguish two cases:
  1. non-overlapping (Podolskij and Vetter (2009)):
     - $\bar{Y}_i$ are uncorrelated as $n \to \infty$.
     - wild bootstrap is valid (Gonçalves, Hounyo and Meddahi (2014, JFEC)):

$$\bar{Y}_i^* = \bar{Y}_i \cdot \eta_i, \quad \text{where} \quad \eta_i \sim \text{ i.i.d.}$$

  2. overlapping (Jacod et al. (2009)): studied in Hounyo, Gonçalves, and Meddahi (2013).
     - $\bar{Y}_i$ are strongly dependent (k_n-dependent and $k_n = \theta\sqrt{n}$).
     - this invalidates the wild bootstrap.

The blocks of blocks bootstrap

- A natural candidate is the block bootstrap:
  - Politis and Romano (1992), Bühlmann and Künsch (1995): "blocks of blocks bootstrap".
- We show that this method is not valid in our context (unless σ is constant).
- The reason is that $\bar{Y}_i$ are heterogeneous (in addition to being strongly dependent).
- We propose a new bootstrap that combines the block bootstrap with the wild bootstrap.

Bootstrap approach

- the bootstrap statistic is $\widetilde{PRV}_n^* = \frac{1}{k_n}\frac{1}{\psi_2}\sum_{i=0}^{n-k_n+1} \bar{Y}_i^{*2}$.

- we use the distribution of $n^{1/4}\left(\widetilde{PRV}_n^* - E^*\left(\widetilde{PRV}_n^*\right)\right)$ to approximate the quantiles of $n^{1/4}\left(PRV_n - IV\right)$.

  - crucial requirement: the wild blocks of blocks bootstrap captures the correct variance.

The wild blocks of blocks bootstrap

- Let J_n be the # of non-overlapping blocks of size b_n.

- For $j = 1, \ldots, J_n$, we set

$$\bar{Y}^{*2}_{i-1+(j-1)b_n} = \bar{B}_{j+1} + \left(\bar{Y}^{2}_{i-1+(j-1)b_n} - \bar{B}_{j+1}\right)\eta_j, \text{ for } 1 \leq i \leq b_n,$$

where

$$\eta_j \sim \text{i.i.d. across blocks.}$$

The wild blocks of blocks bootstrap

- Let J_n be the # of non-overlapping blocks of size b_n.
- For $j = 1, \ldots, J_n$, we set

$$\bar{Y}^{*2}_{i-1+(j-1)b_n} = \bar{B}_{j+1} + \left(\bar{Y}^{2}_{i-1+(j-1)b_n} - \bar{B}_{j+1}\right)\eta_j, \text{ for } 1 \leq i \leq b_n,$$

where

$$\eta_j \sim \text{i.i.d. across blocks.}$$

- This resembles the wild bootstrap of Liu (1988):
  - we use $b_n > 1$ to account for the k_n-dependence of $\bar{Y}_i^2$.
  - we use a different centering: $\bar{B}_{j+1}$ instead of $\widetilde{PRV}_n$.

Finite sample properties of wild blocks of blocks bootstrap

Coverage rates of nominal 95% intervals

Frequency	n	$\xi^2 = 0.0001$		$\xi^2 = 0.01$	
		CLT	boot	CLT	boot
		SV1F			
"2-min"	195	90.89	91.02	83.11	88.51
"15-sec"	1560	93.86	94.01	87.97	93.10
"1-sec"	23400	94.80	94.93	92.87	94.85
		SV2F			
"2-min"	195	88.60	90.09	80.96	87.79
"15-sec"	1560	92.62	93.71	88.02	93.61
"1-sec"	23400	94.68	95.10	92.88	95.12

Based on Jacod et al. (2009) with $\theta = 1/3$ and $k_n = \theta\sqrt{n}$.
$\xi^2 = \omega^2/\sqrt{IQ}$; b_n is chosen by Politis, Romano and Wolf (1999)

Bootstrapping high frequency jump tests

- Suppose now

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + J_t,$$

  - J_t is a jump process.
  - for simplicity, we set $\epsilon_t = 0$ (no noise).

- **Main goal**: to propose bootstrap tests for “no jumps”, i.e.

$$\begin{aligned} H_0 &: \omega \in \Omega_0 \equiv \{\omega : t \longmapsto X_t(\omega) \text{ is continuous on } [0,1]\} \\ H_1 &: \omega \in \Omega_1 \equiv \{\omega : t \longmapsto X_t(\omega) \text{ is discontinuous on } [0,1]\}. \end{aligned}$$

Barndorff-Nielsen and Shephard (2004, 2006)

- Under H_0,

$$T_n = \frac{\sqrt{n}\left(RV_n - BV_n\right)}{\sqrt{\hat{V}_n}} \xrightarrow{d} N\left(0,1\right),$$

$$\begin{aligned} RV_n &= \sum_{i=1}^{n} r_i^2, \\ BV_n &= \frac{1}{k_{1,1}^2}\sum_{i=2}^{n}\left|r_{i-1}\right|\left|r_i\right|, \\ \hat{V}_n &= \left(\theta-2\right)\widehat{IQ}_n = \left(\theta-2\right)\frac{n}{k_{1,\frac{4}{3}}^3}\sum_{i=3}^{n}\left|r_i\right|^{4/3}\left|r_{i-1}\right|^{4/3}\left|r_{i-2}\right|^{4/3}, \end{aligned}$$

  - for any $q \geq 0$, $k_{1,q} = E\left|Z\right|^q$, where $Z \sim N\left(0,1\right)$.
  - $\theta = \left(k_{1,1}^{-4} - 1\right) + 2\left(k_{1,1}^{-2} - 1\right) \approx 2.6090.$

Properties of T_n under H_1

- Under H_1,

$$RV_n - BV_n \xrightarrow{P} \sum_{s \leq 1} (\Delta J_s)^2 > 0.$$

- If $\widehat{IQ}_n$ is robust to jumps, then

$$T_n = O_P\left(\sqrt{n}\right) \to +\infty.$$

- One-sided test: reject H_0 when

$$T_n \geq z_{1-\alpha}.$$

  - this test has asymptotic correct size and is alternative-consistent.
- **Our goal:** to replace $z_{1-\alpha}$ with a bootstrap critical value.

The bootstrap DGP

- We follow Hounyo (2013) and let

$$r_i^* = \sqrt{\hat{v}_i^n} \cdot \eta_i, \qquad \eta_i \perp r_i$$

  - $\eta_i \sim$ i.i.d. $N(0,1)$.
  - $\hat{v}_i^n$ is a (local) estimate of $v_i^n = \int_{(i-1)/n}^{i/n} \sigma_s^2 ds$.
- Crucial feature:
  - r_i^* are locally Gaussian and therefore have no jumps.
  - it satisfies one of the "Golden rules of bootstrap testing" (Davidson and MacKinnon, 1999).
- How should we choose $\hat{v}_i^n$?

Possible choices of $\hat{v}_i^n$

- $\hat{v}_i^n = r_i^2$ (wild bootstrap, as in GM, 2009).
- $\hat{v}_i^n$ is a local RV_n estimate (Hounyo, 2013).
- Here: we allow for a general class of $\hat{v}_i^n$.
- We give a set of conditions on $\{\hat{v}_i^n\}$ such that *any* bootstrap test is valid when testing for jumps.
  - asymptotic validity: the bootstrap controls size and is consistent under H_1.

Bootstrap test statistic

- the bootstrap analogue of T_n is

$$T_n^* = \frac{\sqrt{n}\left(RV_n^* - BV_n^* - E^*\left(RV_n^* - BV_n^*\right)\right)}{\sqrt{\hat{V}_n^*}}.$$

- $\hat{V}_n^*$ is an estimator of

$$V_n^* = Var^*\left(\sqrt{n}\left(RV_n^* - BV_n^*\right)\right).$$

Bootstrap asymptotic validity

- We give high level conditions on $\hat{v}_i^n$ such that under both H_0 *and* H_1,

$$T_n^* \xrightarrow{d^*} N(0,1).$$

- Since

$$\begin{cases} T_n \xrightarrow{st} N(0,1) & \text{under } H_0 \\ T_n \xrightarrow{P} +\infty & \text{under } H_1 \end{cases}$$

  this ensures that the bootstrap test has correct asymptotic size and is alternative-consistent.

- Crucial requirement for bootstrap CLT under H_1:
  - V_n^* should be robust to jumps.

Example 1: local RV_n estimate

- Let $M \geq 1$. For $j = 1, \ldots, n/M$, set

$$\hat{v}^n_{i+(j-1)M} = \frac{1}{M} \sum_{\ell=(j-1)M+1}^{jM} r_\ell^2 \equiv \bar{R}_j, \quad i = 1, \ldots, M$$

- When $M = 1$, we get the regular wild bootstrap.

Example 2: local multipower estimate

- For $M \geq 1$, $L \geq 1$, and $\{p_l : l = 1, \ldots, L\}$ such that $\sum_{l=1}^{L} p_l = 1$, where $p_l \geq 0$.
- For $j = L, \ldots, n/M$,

$$\hat{v}^n_{i+(j-1)M} = \bar{R}_j^{p_1} \cdot \bar{R}_{j-1}^{p_2} \cdot \ldots \cdot \bar{R}_{j-L+1}^{p_L}, \qquad i = 1, \ldots, M.$$

- When $L = 1$, this is the previous example.
- When $M = 1$, we obtain

$$\hat{v}^n_i = |r_i|^{2p_1} |r_{i-1}|^{2p_2} \ldots |r_{i-L+1}|^{2p_L}.$$

General form of $V_n^* = Var^* \left(\sqrt{n} \left(RV_n^* - BV_n^* \right) \right)$

- V_n^* is a function of $\Sigma_n^* = Var^* \left(\sqrt{n} \left(RV_n^*, BV_n^* \right)' \right)$.
- We can show that

$$
\begin{aligned}
\Sigma_{n,1,1}^* &= 2n \sum_{i=1}^{n} \left(\hat{v}_i^n \right)^2 \\
\Sigma_{n,2,2}^* &= \left(k_{1,1}^{-4} - 1 \right) n \sum_{i=2}^{n} \left(\hat{v}_i^n \right) \left(\hat{v}_{i-1}^n \right) \\
&\quad + 2 \left(k_{1,1}^{-2} - 1 \right) n \sum_{i=3}^{n} \left(\hat{v}_i^n \right)^{1/2} \left(\hat{v}_{i-1}^n \right) \left(\hat{v}_{i-2}^n \right)^{1/2} \\
\Sigma_{n,1,2}^* &= n \sum_{i=2}^{n} \left(\hat{v}_i^n \right)^{3/2} \left(\hat{v}_{i-1}^n \right)^{1/2} + n \sum_{i=2}^{n} \left(\hat{v}_i^n \right)^{1/2} \left(\hat{v}_{i-1}^n \right)^{3/2}.
\end{aligned}
$$

The form of V_n^* in Example 1

- Suppose $M = 1$, which implies that $\hat{v}_i^n = r_i^2$. Then,

$$\Sigma_{n,1,1}^* \equiv Var^* \left(\sqrt{n} RV_n^* \right) = 2n \sum_{i=1}^{n} \left(\hat{v}_i^n \right)^2 = 2n \sum_{i=1}^{n} r_i^4.$$

  - not robust to jumps.

The form of V_n^* in Example 1

- Suppose $M = 1$, which implies that $\hat{v}_i^n = r_i^2$. Then,

$$\Sigma_{n,1,1}^* \equiv Var^*\left(\sqrt{n}RV_n^*\right) = 2n\sum_{i=1}^{n}\left(\hat{v}_i^n\right)^2 = 2n\sum_{i=1}^{n} r_i^4.$$

  - not robust to jumps.

- Similar result for $M > 1$:
  - $\Sigma_{n,1,1}^*$ (and hence V_n^*) depends on the efficient blocked quarticity (Mykland, Shephard and Sheppard, 2012).
  - not robust to jumps either.

Implications for local RV_n-based Gaussian bootstrap

- Bootstrap CLT only holds under H_0 (but not under H_1).
- This bootstrap controls size, but is not necessarily alternative-consistent.
- When $M = 1$, $T_n^* \xrightarrow{P^*} +\infty$ under H_1 at the *same rate* as T_n.
  - bootstrap test may lack power (confirmed by our simulations).

The form of V_n^* in Example 2

- Take $M = 1$ for simplicity:

$$\hat{v}_i^n = \left|r_i\right|^{2p_1} \left|r_{i-1}\right|^{2p_2} \ldots \left|r_{i-L+1}\right|^{2p_L}.$$

- Then

$$\Sigma^*_{n,1,1} = 2n \sum_{i=1}^{n} \left(\hat{v}_i^n\right)^2 = 2n \sum_{i=1}^{n} \left|r_i\right|^{4p_1} \left|r_{i-1}\right|^{4p_2} \ldots \left|r_{i-L+1}\right|^{4p_L}.$$

The form of V_n^* in Example 2

- Take $M = 1$ for simplicity:

$$\hat{v}_i^n = \left|r_i\right|^{2p_1} \left|r_{i-1}\right|^{2p_2} \ldots \left|r_{i-L+1}\right|^{2p_L}.$$

- Then

$$\Sigma_{n,1,1}^* = 2n \sum_{i=1}^{n} \left(\hat{v}_i^n\right)^2 = 2n \sum_{i=1}^{n} \left|r_i\right|^{4p_1} \left|r_{i-1}\right|^{4p_2} \ldots \left|r_{i-L+1}\right|^{4p_L}.$$

  - robust to jumps if

$$\max_{l=1,\ldots,L} \left(4p_l\right) < 2 \Longleftrightarrow L > 2 \text{ when } p_l = \frac{1}{L} \text{ for all } l = 1, \ldots, L.$$

- Similar results hold when $M > 1$:
  - Σ_n^* depends on linear combinations of blocked multipower variations of r_i with maximum exponent given by $4p_l$.

Monte Carlo simulations

- $X_t = Y_t + J_t$
- Y_t is given by the SV2F model (Chernov et al., 2003, Huang and Tauchen, 2005).
- J_t is a compound Poisson process with constant jump intensity λ and random jump size distributed as $N(0, \sigma^2_{jmp})$.
  - We set $\sigma^2_{jmp} = 0$ under H_0.
  - We set $\lambda = 0.058$ and $\sigma^2_{jmp} = 1.7241$ under $H_1 \Rightarrow$ jumps account for 10% of total variation in prices.
  - We consider other specifications in the paper.

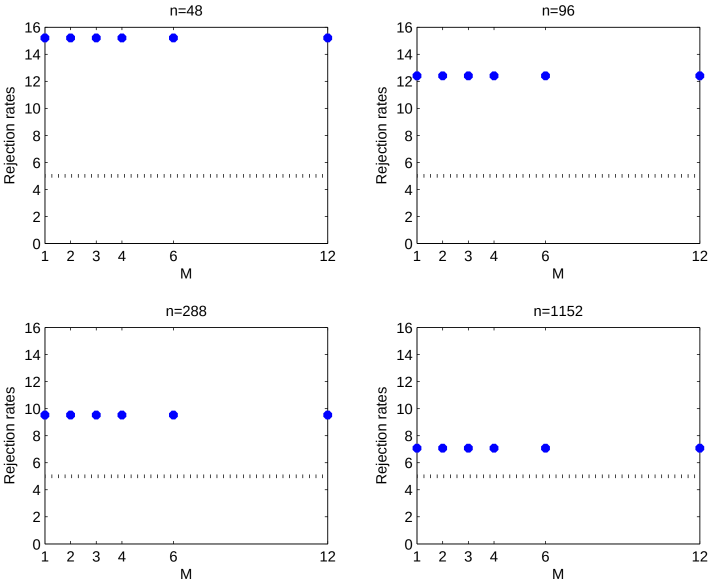


Figure: SV2F model, linear test, no jumps

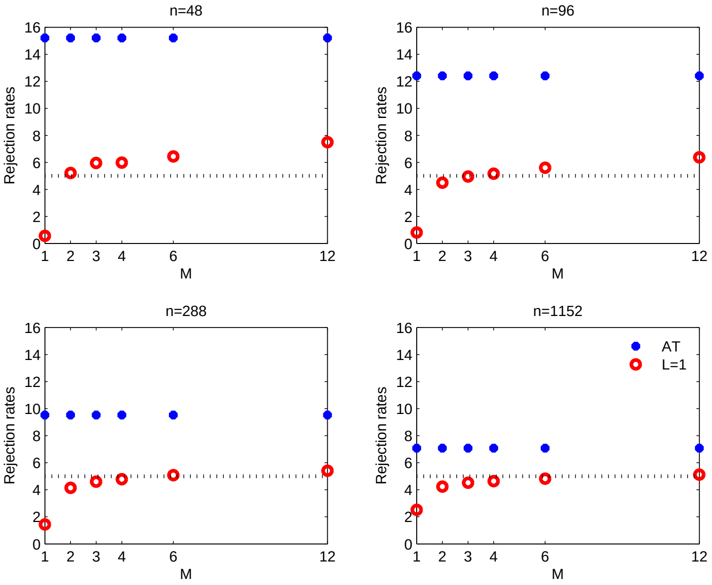


Figure: SV2F model, linear test, no jumps

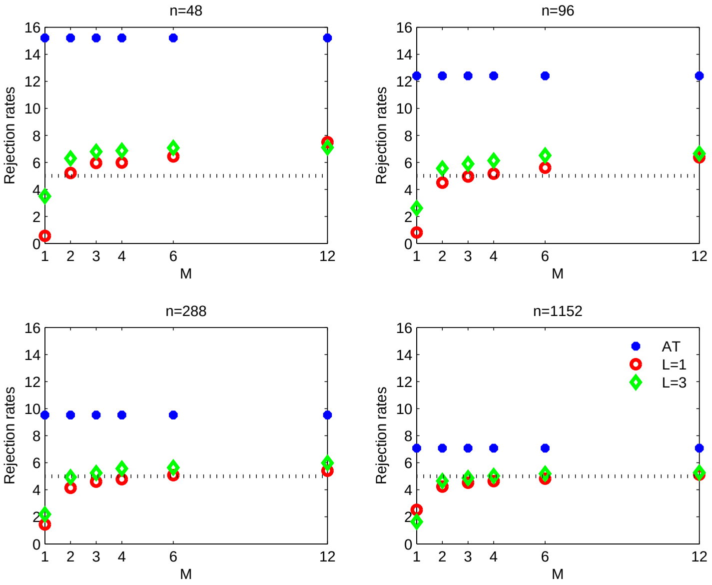


Figure: SV2F model, linear test, no jumps

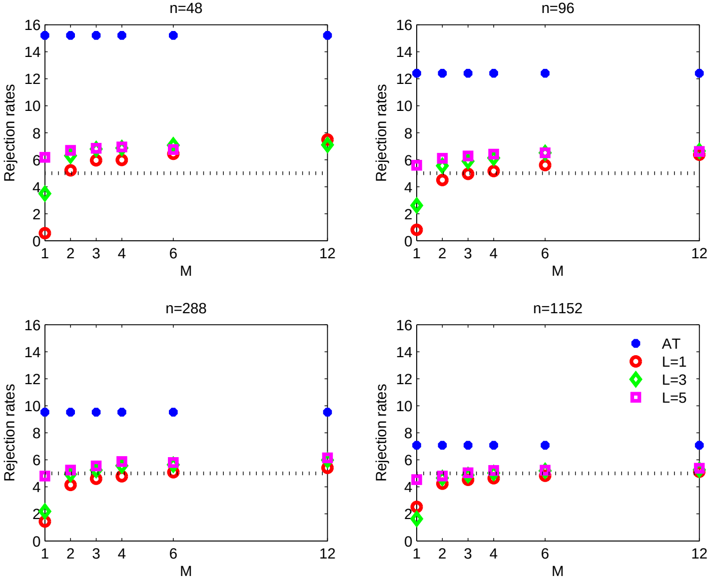


Figure: SV2F model, linear test, no jumps

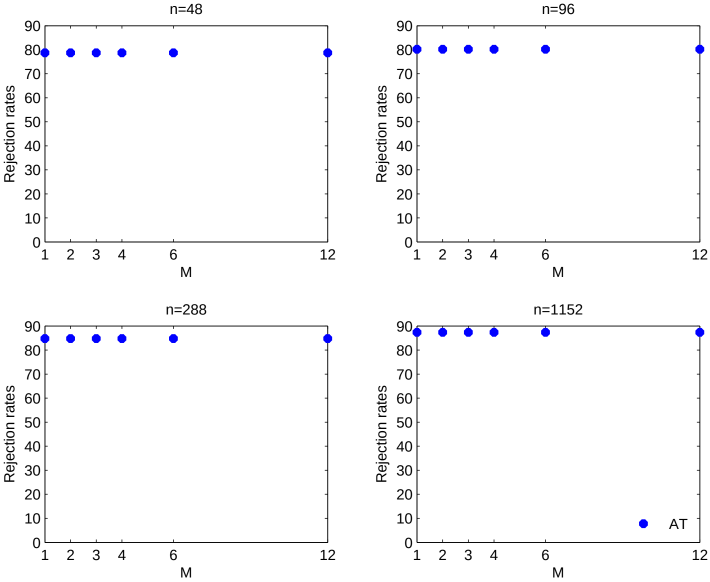


Figure: SV2F model, linear test, power

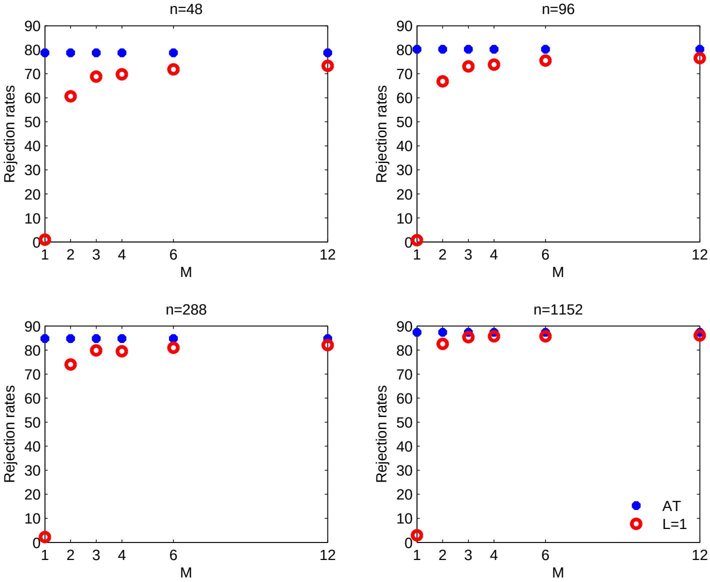


Figure: SV2F model, linear test, power

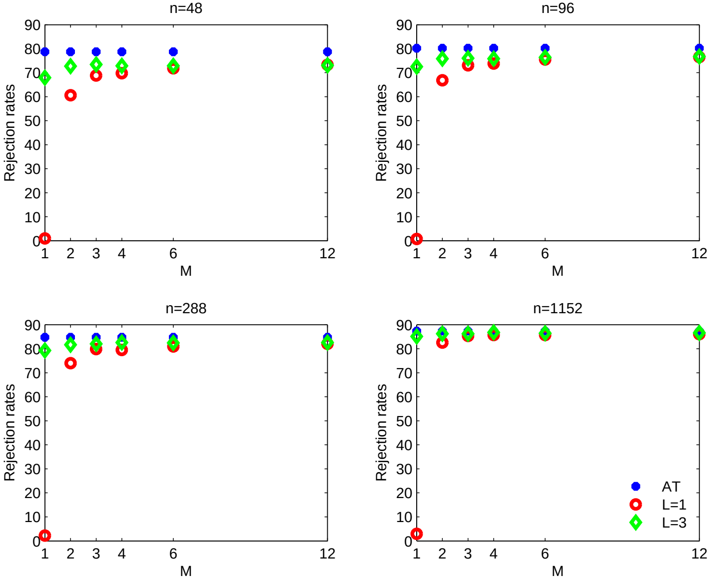


Figure: SV2F model, linear test, power

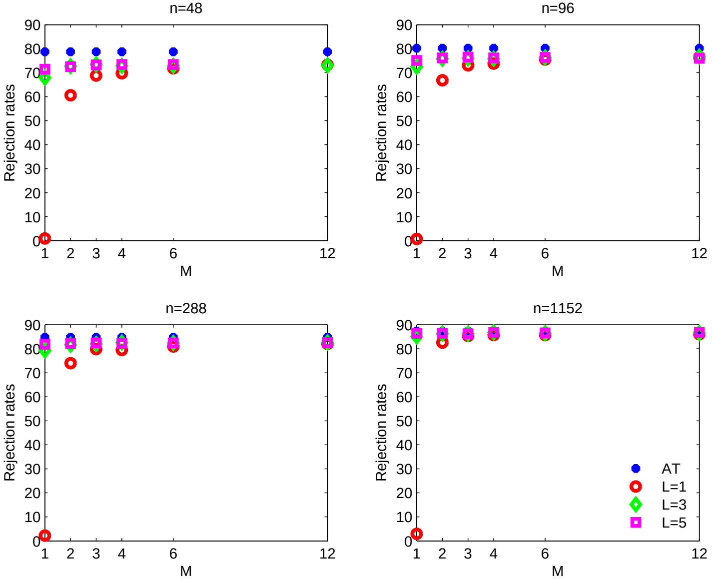


Figure: SV2F model, linear test, power

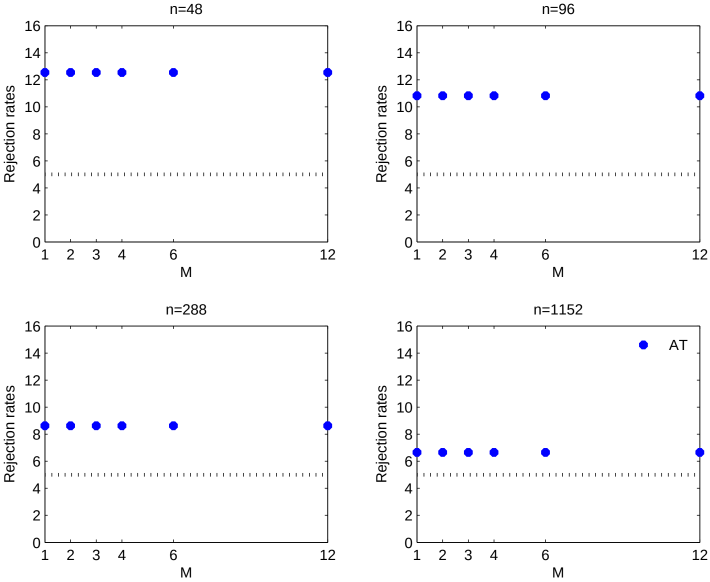


Figure: SV2F model, log test, no jumps

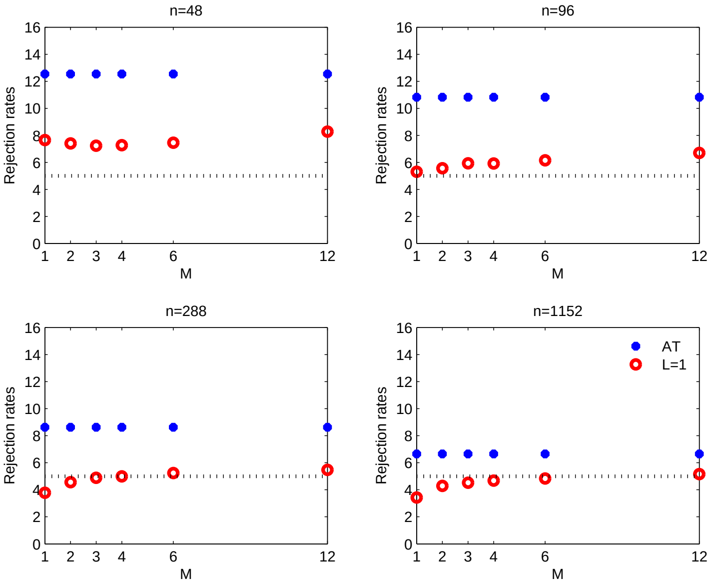


Figure: SV2F model, log test, no jumps

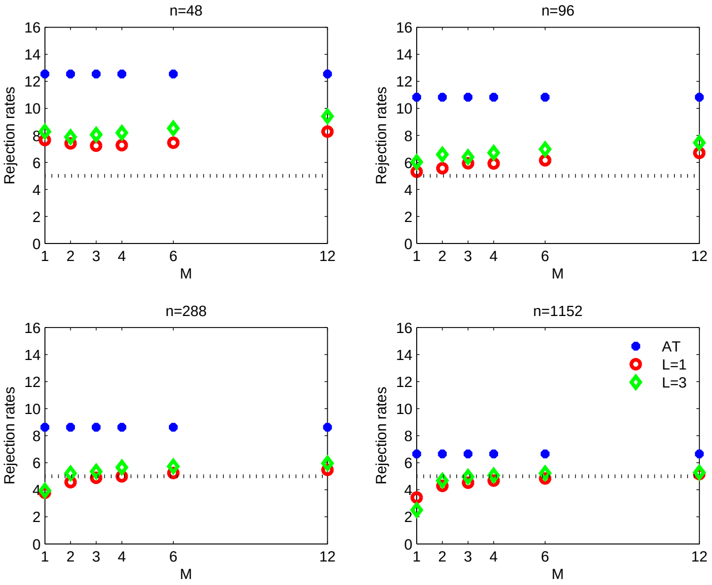


Figure: SV2F model, log test, no jumps

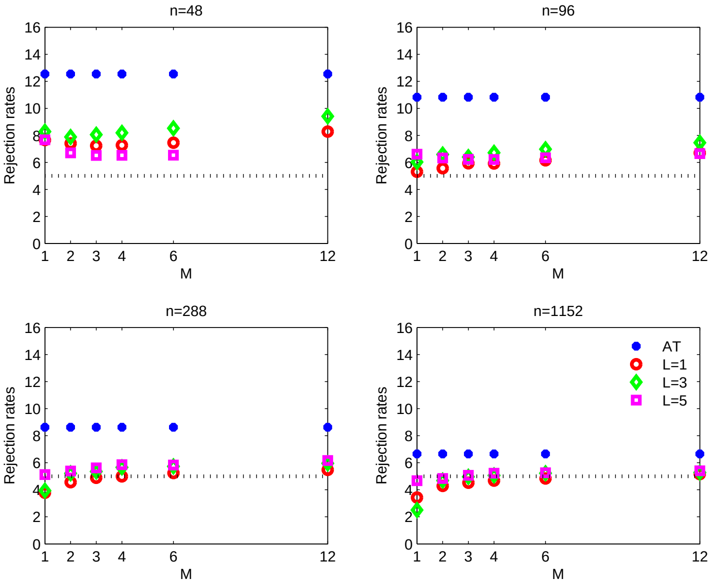


Figure: SV2F model, log test, no jumps

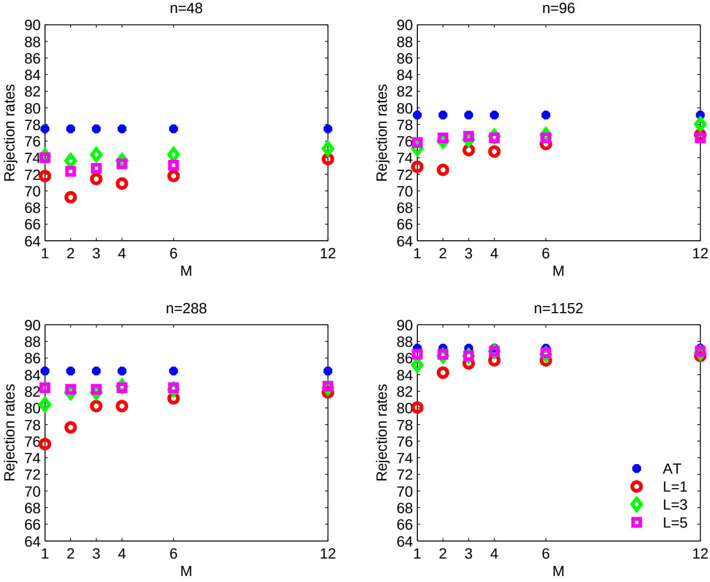


Figure: SV2F model, log test, power

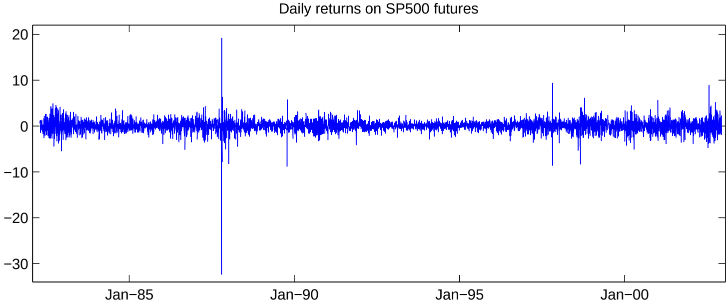


Figure: Sample period: April 21, 1982 - December 9, 2002

% of days identified as jumps by the daily statistics for the S&P 500 futures index

Sample period: April 21, 1982 - December 9, 2002.

Linear version of T_n							
		M					
		1	2	3	4	6	12
BN-S	21.0						
$L = 1$		2.5	9.1	10.3	10.7	11.8	12.7
$L = 3$		3.4	11.2	11.8	12.6	12.5	12.4
$L = 5$		6.0	11.6	12.3	12.6	12.3	9.4
Log version of T_n							
BN-S	17.9						
$L = 1$		12.1	12.6	12.7	13.1	13.7	14.2
$L = 3$		9.1	13.5	13.9	14.3	14.4	14.7
$L = 5$		6.9	11.8	11.9	11.8	11.8	9.5

Based on 5-min returns on the S&P 500 futures ($n = 80$)

Nominal level: 5%

Conclusions and possible extensions

- Bootstrapping high frequency data is challenging:
  - one challenge is the heterogeneity induced by stochastic volatility.
  - another challenge is the serial dependence created by market microstructure noise.
- The wild blocks of blocks bootstrap can handle both features.
- We also discussed bootstrap tests for the null "no jumps" (under no noise).
  - the local Gaussian bootstrap can be used provided we choose $\hat{v}_i^n$ appropriately.
- Things to do:
  - how to select L and M ?
  - how to bootstrap jump tests under market microstructure noise?

Monte Carlo simulations

- SV1F model (Heston, 1993)

$$
\begin{aligned}
dX_t &= \left(0.05/252 - \nu_t/2\right) dt + \sigma_t dB_t, \\
d\nu_t &= 5/252 \left(0.04/252 - \nu_t\right) dt + 0.05/252 \left(\nu_t\right)^{1/2} dW_t, \\
\nu_t &= \sigma_t^2, Corr(B, W) = -0.5.
\end{aligned}
$$

- SV2F model (Chernov et al, 2003, Huang and Tauchen, 2005):

$$
\begin{aligned}
dX_t &= 0.03dt + \sigma_t dW_t, \\
\sigma_t &= s\text{-}exp\left(-1.2 + 0.04\tau_{1t} + 1.5\tau_{2t}\right), \\
d\tau_{1t} &= -0.00137\tau_{1t}dt + dB_{1t}, \\
d\tau_{2t} &= -1.386\tau_{2t}dt + \left(1 + 0.25\tau_{2t}\right) dB_{2t}, \\
corr\left(dW_t, dB_{1t}\right) &= corr\left(dW_t, dB_{2t}\right) = -0.3.
\end{aligned}
$$

Bootstrap intervals

- We show that

$$S_n^* \equiv n^{1/4}\left(\widetilde{PRV}_n^* - E^*\left(\widetilde{PRV}_n^*\right)\right) \to^{d^*} N\left(0, V\right),$$

in prob-P.

- Since $S_n \equiv n^{1/4}\left(PRV_n - IV\right) \to^{st} N\left(0, V\right)$, this justifies percentile-type intervals for IV :

$$IC_{perc,0.95} = PRV_n \pm p_{0.95}^* \frac{1}{n^{1/4}},$$

  - $p_{0.95}^*$ is the 95% quantile of the bootstrap distribution of $|S_n^*|$.

Overview of our results

- We verify our high level conditions for Examples 1 and 2.
- For Example 1, our conditions are verified under H_0 (but not under H_1).
  - this bootstrap controls size, but is not necessarily alternative-consistent.
  - for $L = M = 1$ (regular wild bootstrap), $T_n^* \xrightarrow{P^*} +\infty$ at the *same rate* as T_n.

Overview of our results

- We verify our high level conditions for Examples 1 and 2.
- For Example 1, our conditions are verified under H_0 (but not under H_1).
  - this bootstrap controls size, but is not necessarily alternative-consistent.
  - for $L = M = 1$ (regular wild bootstrap), $T_n^* \xrightarrow{P^*} +\infty$ at the *same rate* as T_n.
- Example 2 verifies our conditions under both H_0 and H_1 if

$$\max_{l=1,\dots,L} (p_l) < \frac{1}{2},$$

or equivalently

$$L > 2 \text{ when } p_l = \frac{1}{L} \text{ for all } l = 1, \dots, L.$$