The Long and the Short of the Risk-Return Trade-Off *

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Abstract

The risk-return trade-off is so central to financial economics that the debate over its existence and magnitude is never ending. Researchers have looked for it at high- and low-frequency and have run many regressions to relate volatility measures to future returns. This paper has a high ambition. It aims at finding a model that reproduces a large collection of stylized facts regarding the risk-return trade-off at short and long horizons. Writing the model in Bonomo et al. (2011) at the daily frequency, that is a long-run risk model with volatility risk and generalized disappointment aversion preferences, and keeping the same calibration, we are able to reproduce the first and second moments of the equity premium and the risk-free rate, the first moment of the variance premium and realized volatility, the predictability of returns by the dividend ratio, the long-run predictability of cumulative returns by the past cumulative variance, the short-run predictability of returns by the variance premium, and last but not least the daily autocorrelation patterns at many lags of the VIX and of the variance premium, as well as the daily cross-correlations of these two measures with leads and lags of daily returns.

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1 Introduction

The existence of a systematic trade-off between risk and expected returns is central to modern finance, but the disappointing fact is that after more than two decades of empirical research studies do not agree even on the sign of the linear relationship between the equity premium and the conditional stock market volatility. Some find a positive relation, others a negative one. In some studies, there is no significant trade-off.

Several explanations are proposed for this lingering debate. In a survey about measuring and modeling variation in the risk-return trade-off, Lettau and Ludvigson (2010) attribute in large part the disagreement in the empirical literature on this relation to the limited amount of information generally used to model the conditional mean and conditional volatility of excess stock market returns. Therefore, the model does not reflect the potentially richer information set of the market participants and the relation between risk and return is misspecified. Recently, Rossi and Timmermann (2010) propose a different explanation. They argue that there is no theoretical reason for assuming a linear relationship between the expected returns and the conditional volatility and adopt a regression-tree approach to carve out the state space through a sequence of piece-wise constant models that approximate the unknown shape of the risk-return relation. They found support for nonlinear patterns in the risk-return trade-off. Another reason for not finding a significant and stable relation may be the noise-signal ratio. Bandi and Perron (2008) find that the dependence is statistically mild at short horizons, which explains the contradicting results in the literature, but increases with the horizon and is strong in the long run (between 6 and 10 years). Another more structural approach is proposed by Bollerslev and Zhou (2005). They provide a theoretical framework for assessing the empirical links between returns and realized volatilities. They show that the sign of the correlation between contemporaneous return and realized volatility depends importantly on the underlying structural parameters that enter nonlinearly in the coefficient.

In this paper, we address these various lines of explanation by relying on an equilibrium consumption-based asset pricing model with long-run risk in volatility as in Bonomo et al. (2011)[1].

[1] Bonomo et al. (2011) propose an asset pricing model with generalized disappointment aversion preferences and long-run volatility risk. With Markov switching fundamentals, they derive closed-form solutions for all returns moments and predictability regressions. The model produces first and second moments of price-dividend ratios and asset returns as well as return predictability patterns in line with the data. The same long-run risk model can be used to reproduce stylized facts related to the risk-return trade-off. Bonomo et al. (2011) extends considerably the literature on closed-form pricing formulas when one assumes a Markov-switching model like, for instance, Cecchetti

In this model, where agents are disappointment averse, the relation between expected returns and conditional volatility is certainly not linear, so we can produce patterns that are consistent with the empirical shapes found in Rossi and Timmermann (2010). The model being a long-run risk model, we can also hope to reproduce the short- and long-run patterns exhibited in Bandi and Perron (2008). Moreover, the model will produce endogenously a predictor such as the consumption-aggregate wealth ratio (cay), which according to Lettau and Ludvigson (2010) is a strong predictor of both returns and volatility of the stock market.

A recent trend in the literature has also put forward the variance risk premium as a strong predictor of stock returns in the short run. The variance risk premium is measured as the difference between the squared VIX index and expected realized variance. Bollerslev et al. (2009) and Drechsler and Yaron (2011) provide a rationalization of this predictability by the variance premium based on extensions of the Bansal and Yaron (2004) long-run risk model. Both models add a time-varying volatility of volatility to the initial model where it was constant. A serious limitation of both models is that the model-implied measure of the variance premium is not based on an accumulation of daily quantities as in the data but on monthly conditional measures. Moreover, they do not consider the long-run risk-return trade-off.

Our main contribution is to reproduce relations between returns and volatility at short and long horizons with an equilibrium model calibrated at a high-frequency daily level. We can then solve the model daily and construct realized quantities at lower frequencies. A key advantage of the model proposed by Bonomo et al. (2011) is to find analytical formulas for moments of asset pricing quantities such as payoff ratios and returns, and for coefficients of predictability regressions at any horizon. In this framework we are able to produce results at high frequency together with the long-run regressions of Bandi and Perron (2008) in the same model.

At the high-frequency level, we assess the capacity of the model to reproduce the autocorrelation of VIX^2 and of the variance premium, and cross-correlations of these quantities at various leads and lags. We also build measures of monthly realized volatility by summing daily squared returns and compute moments of realized volatility. The variance premium is obtained by first taking the expectation under the risk-neutral measure of this realized volatility and subtracting the latter from the obtained risk-neutral volatility. The predictability of returns by the variance premium

et al. (1990), Bonomo and Garcia (1994), and Calvet and Fisher (2007).

is then established for horizons of one to twelve months. At longer horizons, we reproduce the predictability regressions of Bandi and Perron (2008) by aggregating returns and volatilities over periods of one to ten years.

We use a generalized disappointment aversion model which is identical in many dimensions to Bonomo et al. (2011). Preference parameters are kept the same and expected consumption growth is constant. Therefore, the only long-run risk is an economic uncertainty risk captured by the volatility of consumption. To capture a richer short-run dynamics in volatility and the predictability of returns by the variance premium we add a volatility factor that exhibits less persistence than in the original model. It is in the spirit of Bollerslev et al. (2009) who have also a long-run risk model of volatility only. In Drechsler and Yaron (2011), jump processes are added to both expected consumption growth and consumption volatility to obtain similar effects. Given this calibration, we obviously maintain the results we had in Bonomo et al. (2011) in terms of matching the first and second asset pricing moments and the predictability of returns by the dividend price ratio.

In terms of moments of the VIX^2, RV and VRP, we match the mean of the VIX^2 but tend to overestimate the mean of the realized volatility, therefore underestimating the mean of the variance premium. For the second moments, we overestimate the standard deviation of both VIX^2 and RV and underestimate the standard deviation of the variance premium. For the short-run risk-return trade-off stylized facts, we are able to reproduce the daily autocorrelation patterns in VIX^2 and VRP, up to 90 lags, that is the more persistent autocorrelation for the first measure and the faster decay for the variance premium. For the cross-correlations of VIX^2 and VRP with 22 leads and lags of daily returns, we observe a negative pattern in the lags and a close to zero pattern in the leads for both measures. Our model reproduces well the negative cross-correlations in the lags (interpreted in the literature as a leverage effect) but overestimates the positive cross-correlations in the leads. Therefore, our model creates a stronger volatility feedback effect than observed. This slight short-run predictability remains in the long-run since the model reproduces the increasing explanatory power at longer horizons found by Bandi and Perron (2008). For the short-run predictability of returns by the variance premium, we obtain with the model the pattern exhibited by the data (a peak around 2 to 3 months and a slow decline up to 12 months).

The rest of the paper is organized as follows. Section 2 sets up the model for both preferences and dynamics of fundamentals and provides the asset pricing solution. In Section 3, we detail the various measures used as stylized facts for the risk-return trade-off and provide the model-based analytical formulas for assessing the trade-off. Section 4 reviews the empirical stylized facts for these various measures over the period 1990-2012 for facts involving the variance risk premium and 1930-2012 for the long-run risk-returns trade-off. We also compute the short-run measures over the period 1990-2007 to account for the potential effect of the crisis on these quantities. The calibration and the assessment of the model along the various measures of the risk-return trade-off are reported in Section 5. Section 6 concludes. An appendix provides the details of the analytical derivations for the asset pricing moments and the risk-return trade-off measures.

2 Model Setup, Assumptions and Asset Pricing Solution

We assume that there are $1/\Delta$ trading periods in a month, and that month t contains the periods $t-1+j\Delta, j=1,2,\ldots,1/\Delta$. For example, $\Delta = 1/22$ for daily periods and $\Delta = 1/(78\times 22)$ for 5-min interval periods. We refer to the month as the frequency 1 and to the period as the frequency Δ. So defined, the frequency h refers to h months or equivalently h/Δ periods. For example, the frequency 12 corresponds to yearly. We assume that economic agents decision interval corresponds to the frequency Δ so that dynamics of preferences, endowments and other exogenous state variables are given at the frequency Δ.

2.1 Equilibrium Consumption and Dividends Growths Dynamics

We assume that equilibrium consumption and dividends growths are unpredictable, at least at the frequency Δ, and that their conditional variance as well as their conditional correlation change according to a Markov variable s_t which takes N values, $s_t \in \{1,2,\ldots,N\}$, when N states of nature are assumed for the economy. The sequence s_t evolves according to a transition probability matrix P defined as:

$$P^\top = [p_{ij}]_{1\le i,j\le N} \quad \text{and} \quad p_{ij} = Prob\left(s_{t+\Delta} = j \mid s_t = i\right). \tag{1}$$

Let $\zeta_t = e_{s_t}$, where e_j is the $N\times 1$ vector with all components equal to zero but the jth component is equal to one. Therefore, the dynamics of consumption and dividends are given by:

$$\begin{aligned} g_{c,t+\Delta} &= \ln\left(\frac{C_{t+\Delta}}{C_t}\right) = \mu_x + \sigma_t \varepsilon_{c,t+\Delta} \\ g_{d,t+\Delta} &= \ln\left(\frac{D_{t+\Delta}}{D_t}\right) = \mu_x + \nu_d \sigma_t \varepsilon_{d,t+\Delta} \end{aligned} \tag{2}$$

where $\sigma_t^2 = \omega_c^\top \zeta_t$, and where

$$\begin{pmatrix} \varepsilon_{c,t+\Delta} \\ \varepsilon_{d,t+\Delta} \end{pmatrix} \mid \left\{ \varepsilon_{c,j\Delta}, \varepsilon_{d,j\Delta}, j \leq \frac{t}{\Delta}; \zeta_{k\Delta}, k \in \mathbb{Z} \right\} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_t \\ \rho_t & 1 \end{pmatrix} \right), \tag{3}$$

with

$$\rho_t = \frac{1 - \exp\left(-\beta_{\rho 0} - \beta_{\rho\sigma} \ln \sigma_t^2\right)}{1 + \exp\left(-\beta_{\rho 0} - \beta_{\rho\sigma} \ln \sigma_t^2\right)} = \rho^\top \zeta_t. \tag{4}$$

The scalar μ_x is the expected growth of aggregate consumption, which is assumed equal to that of aggregate dividends. The two vectors ω_c and ρ contain state values of the volatility of consumption growth and of the correlation between consumption growth and dividend growth, respectively. The ith element of a vector refers to the value in state $s_t = i$. Equation (4) shows that the conditional correlation between consumption and dividends growths depends on the state of the economy as determined by the volatility of aggregate consumption. In particular if $\beta_{\rho\sigma} = 0$, then consumption and dividends growth correlation is constant; if $\beta_{\rho\sigma} > 0$ this correlation increases with macroeconomic uncertainty.

We assume that the log conditional variance of aggregate consumption growth is given by

$$\ln \sigma_t^2 = a_z + b_{1z} z_{1,t} + b_{2z} z_{2,t} \tag{5}$$

where $z_{1,t}$ and $z_{2,t}$ are two independent two-state Markov chains that can take values 0 and 1, corresponding to a low (L) and a high (H) states. The chain $z_{i,t}$ has the persistence ϕ_{iz}, a nonnegative skewness and the kurtosis k_{iz}. Following Bonomo et al. (2011) its transition matrix P_{iz} may be written

$$P_{iz}^{\top} = \begin{pmatrix} p_{iz,LL} & 1-p_{iz,LL} \\ 1-p_{iz,HH} & p_{iz,HH} \end{pmatrix} \text{ with conditional state probabilities given by}$$
$$p_{iz,LL} = \frac{1+\phi_{iz}}{2} + \frac{1-\phi_{iz}}{2}\sqrt{\frac{k_{iz}-1}{k_{iz}+3}} \text{ and } p_{iz,HH} = \frac{1+\phi_{iz}}{2} - \frac{1-\phi_{iz}}{2}\sqrt{\frac{k_{iz}-1}{k_{iz}+3}}. \tag{6}$$

The kurtosis of the two-state Markov chain $z_{i,t}$ fully characterizes its stationary distribution, for which the unconditional state probabilities are given by

$$\begin{aligned} \pi_{iz,L} &= P\left(z_{i,t}=0\right) = \frac{1-p_{iz,HH}}{2-p_{iz,LL}-p_{iz,HH}} = \frac{1}{2}+\frac{1}{2}\sqrt{\frac{k_{iz}-1}{k_{iz}+3}} \\ \pi_{iz,H} &= P\left(z_{i,t}=1\right) = \frac{1-p_{iz,LL}}{2-p_{iz,LL}-p_{iz,HH}} = \frac{1}{2}-\frac{1}{2}\sqrt{\frac{k_{iz}-1}{k_{iz}+3}}, \end{aligned} \tag{7}$$

and the skewness is given by

$$s_{iz} = \frac{\pi_{iz,L}-\pi_{iz,H}}{\sqrt{\pi_{iz,L}\pi_{iz,H}}} = \sqrt{k_{iz}-1}. \tag{8}$$

The combination of the two states of $z_{1,t}$ and the two states of $z_{2,t}$ leads to four distinct states for the economy: $LL \equiv 1$, $LH \equiv 2$, $HL \equiv 3$ and $HH \equiv 4$. By the independence of the chains $z_{1,t}$ and $z_{2,t}$, the transition probability matrix associated with the four states of the economy also derives easily as $P = P_{1z} \otimes P_{2z}$. The state values of the conditional variance of aggregate consumption growth are given by $\omega_c = \begin{pmatrix} \exp\left(a_z\right) & \exp\left(a_z+b_{2z}\right) & \exp\left(a_z+b_{1z}\right) & \exp\left(a_z+b_{1z}+b_{2z}\right) \end{pmatrix}^{\top}$.

The logarithm of conditional variance $\ln \sigma_t^2$ has the mean μ_σ, the volatility σ_σ and the skewness s_σ that we want to match with the coefficients a_z, $b_{1,z}$ and $b_{2,z}$ in equation (5). We also assume that the first component $z_{1,t}$ has a zero skewness ($s_{1z}=0$) which for a two-state Markov chain is also equivalent to a unitary kurtosis ($k_{1z}=1$) and constant conditional volatility (homoscedasticity). Given ϕ_{1z}, ϕ_{2z} and κ_{2z}, we solve for a_z, $b_{1,z}$ and $b_{2,z}$ to match the mean μ_σ, the volatility σ_σ and the skewness s_σ of $\ln \sigma_t^2$. We find that

$$\begin{aligned} b_{1z} &= \frac{\sigma_\sigma}{\sqrt{\pi_{1z,1}\pi_{1z,2}}}\left(1-\left(\frac{s_\sigma}{s_{2z}}\right)^{2/3}\right)^{1/2} \text{ and } b_{2z} = \frac{\sigma_\sigma}{\sqrt{\pi_{2z,1}\pi_{2z,2}}}\left(\frac{s_\sigma}{s_{2z}}\right)^{1/3} \\ a_z &= \mu_\sigma - b_{1z}\pi_{1z,2} - b_{2z}\pi_{2z,2}. \end{aligned} \tag{9}$$

Equation (9) implies that $s_\sigma < s_{2z}$, or equivalently $\kappa_{2z} > 1 + s_\sigma^2$. Later in our calibration analysis, we assume that the first component is very persistent, with $\phi_{1z}^{1/\Delta}$ close to one, and that the second component is not persistent, with $\phi_{2z}^{1/\Delta}$ typically less than 0.9.

2.2 Preferences

The representative investor has generalized disappointment aversion (GDA) preferences of Routledge and Zin (2010). Following Epstein and Zin (1989) and Weil (1989), such an investor derives utility from consumption, recursively as follows:

$$\begin{aligned} V_t &= \left\{ (1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left[\mathcal{R}_t \left(V_{t+\Delta} \right) \right]^{1-\frac{1}{\psi}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \quad \text{if } \psi \neq 1 \\ &= C_t^{1-\delta} \left[\mathcal{R}_t \left(V_{t+\Delta} \right) \right]^{\delta} \quad \text{if } \psi = 1. \end{aligned} \tag{10}$$

The current period lifetime utility V_t is a combination of current consumption C_t, and $\mathcal{R}_t \left(V_{t+\Delta} \right)$, a certainty equivalent of next period lifetime utility. With GDA preferences the risk-adjustment function $\mathcal{R}(\cdot)$ is implicity defined by:

$$\frac{\mathcal{R}^{1-\gamma} - 1}{1-\gamma} = \int_{-\infty}^{\infty} \frac{V^{1-\gamma} - 1}{1-\gamma} dF(V) - \ell \int_{-\infty}^{\theta \mathcal{R}} \left(\frac{(\theta \mathcal{R})^{1-\gamma} - 1}{1-\gamma} - \frac{V^{1-\gamma} - 1}{1-\gamma} \right) dF(V), \tag{11}$$

where $\ell \geq 0$ and $0 < \theta \leq 1$. When ℓ is equal to zero, $\mathcal{R}$ becomes the Kreps and Porteus (1978) preferences, while V_t represents Epstein and Zin (1989) recursive utility. When $\ell > 0$, outcomes lower than $\theta \mathcal{R}$ receive an extra weight ℓ, decreasing the certainty equivalent. Thus, the parameter ℓ is interpreted as a measure of disappointment aversion, while the parameter θ is the percentage of the certainty equivalent $\mathcal{R}$ such that outcomes below it are considered disappointing[2]. Equation (11) makes clear that the probabilities to compute the certainty equivalent are redistributed when disappointment sets in, and that the threshold determining disappointment is changing over time.

With KP preferences, Hansen et al. (2008) derive the stochastic discount factor in terms of the continuation value of utility of consumption, as follows:

$$M_{t,t+\Delta}^{*} = \delta \left(\frac{C_{t+\Delta}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{V_{t+\Delta}}{\mathcal{R}_t \left(V_{t+\Delta} \right)} \right)^{\frac{1}{\psi} - \gamma} = \delta \left(\frac{C_{t+\Delta}}{C_t} \right)^{-\frac{1}{\psi}} Z_{t+\Delta}^{\frac{1}{\psi} - \gamma}, \tag{12}$$

[2]Notice that the certainty equivalent, besides being decreasing in γ, is also decreasing in ℓ (for $\ell \geq 0$), and decreasing in θ (for $0 < \theta \leq 1$). Thus ℓ and θ are also measures of risk aversion, but of different types than γ.

where

$$Z_{t+\Delta} = \frac{V_{t+\Delta}}{\mathcal{R}_t\left(V_{t+\Delta}\right)} = \left(\delta\left(\frac{C_{t+\Delta}}{C_t}\right)^{-\frac{1}{\psi}} R_{c,t+\Delta}\right)^{\frac{1}{1-\frac{1}{\psi}}}, \tag{13}$$

and where the second equality in Eq. (13) implies an equivalent representation of the stochastic discount factor given in Eq. (12), based on consumption growth and the gross return $R_{c,t+\Delta}$ to a claim on future aggregate consumption stream. In general this return is unobservable. The return to a stock market index is sometimes used to proxy for this return as in Epstein and Zin (1991); or other components can be included such as human capital with assigned market or shadow values. If $\gamma = 1/\psi$, Eq. (12) corresponds to the stochastic discount factor of an investor with time-separable utility and constant relative risk aversion, where the powered consumption growth values short-run consumption risk as usually understood. The ratio of future utility $V_{t+\Delta}$ to the certainty equivalent of this future utility $\mathcal{R}_t\left(V_{t+\Delta}\right)$ will add a premium for long-run consumption risk as put forward by Bansal and Yaron (2004) and measured by Hansen et al. (2008).

For GDA preferences, long-run consumption risk enters in an additional term capturing disappointment aversion[3], as follows:

$$M_{t,t+\Delta} = M^*_{t,t+\Delta}\left(\frac{1+\ell I\left(Z_{t+\Delta}<\theta\right)}{1+\ell\theta^{1-\gamma}E_t\left[I\left(Z_{t+\Delta}<\theta\right)\right]}\right), \tag{14}$$

where $I(\cdot)$ is an indicator function that takes the value 1 if the condition is met and 0 otherwise.

2.3 Asset Pricing Solution

In this model, we can solve for asset prices analytically, for example the price-dividend ratio $P_{d,t}/D_t$ (where $P_{d,t}$ is the price of the portfolio that pays off equity dividend), the price-consumption ratio $P_{c,t}/C_t$ (where $P_{c,t}$ is the price of the unobservable portfolio that pays off consumption) and the price $P_{f,t}/1$ of the one-period risk-free bond that delivers one unit of consumption. To obtain these asset prices, we need expressions for $\mathcal{R}_t\left(V_{t+\Delta}\right)/C_t$, the ratio of the certainty equivalent of future lifetime utility to current consumption, and for V_t/C_t, the ratio of lifetime utility to current consumption. The Markov property of the model is crucial for deriving analytical formulas for

[3] Although Routledge and Zin (2010) do not model long-run consumption risk as it is done in Bansal and Yaron (2004), they discuss how its presence could interact with GDA preferences in determining the marginal rate of substitution.

these expressions and we adopt the following notation:

$$\frac{\mathcal{R}_t\left(V_{t+\Delta}\right)}{C_t}=\lambda_{1z}^{\top}\zeta_t,\quad \frac{V_t}{C_t}=\lambda_{1v}^{\top}\zeta_t,\quad \frac{P_{d,t}}{D_t}=\lambda_{1d}^{\top}\zeta_t \text{ and } P_{f,t}=\lambda_{1f}^{\top}\zeta_t. \tag{15}$$

Solving these ratios amounts to characterize the vectors λ_{1z}, λ_{1v}, λ_{1d} and λ_{1f} as functions of the parameters of the consumption and dividends, dynamics and of the recursive utility function defined above. In Appendix A, we provide explicit analytical expressions for these ratios.

We use results from Bonomo et al. (2011) to show that the excess log equity return over the risk-free rate $r_{t+\Delta}$ can also be written as

$$r_{t+\Delta}=\zeta_t^{\top}\Lambda\zeta_{t+\Delta}+\sqrt{\omega_d^{\top}\zeta_t}\varepsilon_{d,t+\Delta}, \tag{16}$$

where the components of matrix Λ are explicitly defined by

$$\nu_{ij}=\ln\left(\frac{\lambda_{1d,j}+1}{\lambda_{1d,i}}\right)+\mu_{d,i}+\ln\lambda_{1f,i}. \tag{17}$$

3 The Risk-Return Trade-off

3.1 Measuring Risk and Reward

Following Bandi and Perron (2008) who examine the predictability of future long-horizon excess returns by past long-horizon realized variance, we define one-period excess log returns and realized variance by

$$r_{t,t+1}=\sum_{j=1}^{1/\Delta}r_{t+j\Delta} \text{ and } \sigma_{t-1,t}^2=\sum_{j=1}^{1/\Delta}r_{t-1+j\Delta}^2, \tag{18}$$

and also aggregate values over multiple periods as

$$r_{t,t+h}=\sum_{l=1}^{h}r_{t+l-1,t+l} \text{ and } \sigma_{t-m,t}^2=\sum_{l=1}^{m}\sigma_{t-l,t-l+1}^2. \tag{19}$$

Notice that in the empirical investigation of Bandi and Perron (2008), monthly excess returns and realized variance are based on daily returns, thus corresponding to $\Delta=1/22$. Furthermore, their realized variance is based on nominal returns and not on real returns, due to the unavailability of daily inflation data. Notice that nominal excess log returns over the log risk-free return are identical to their real counterparts since inflation rate cancels out in the subtraction. To the contrary, we measure realized variance using excess returns as we do not explicitly model inflation. In principle, this would lead to minor differences in empirical studies.

3.2 Analytical Formulas for Assessing the Risk-Return Tradeoff

We consider the following regression:

$$\frac{r_{t,t+h}}{h} = \alpha_{mh} + \beta_{mh} \frac{\sigma^2_{t-m,t}}{m} + \epsilon^{(m)}_{t,t+h}, \tag{20}$$

for which population values of the intercept α_{mh}, the slope coefficient β_{mh} and the coefficient of determination R^2_{mh} are given by

$$\begin{aligned} \alpha_{mh} &= \frac{E\left[r_{t,t+h}\right]}{h} - \beta_{mh} \frac{E\left[\sigma^2_{t-m,t}\right]}{m} \\ \beta_{mh} &= \frac{m}{h} \frac{Cov\left(\sigma^2_{t-m,t}, r_{t,t+h}\right)}{Var\left[\sigma^2_{t-m,t}\right]} \quad \text{and} \quad R^2_{mh} = \frac{Cov\left(\sigma^2_{t-m,t}, r_{t,t+h}\right)^2}{Var\left[\sigma^2_{t-m,t}\right] Var\left[r_{t,t+h}\right]}. \end{aligned} \tag{21}$$

To be consistent with economic theory, the regression coefficient β_{mh} should be positive, so that higher variance leads to higher expected returns. In the context of the reduced-form general equilibrium asset pricing model described in Section 2, we provide analytical formulas for the population values defined in Eq. (21). These quantities are relevant for assessing the risk-return relation through the predictability regression (20).

Expected values, variances and covariances in equation (21) may be expressed in terms of the components of the mean vector and the autocovariance matrices, $\Gamma^X(l)$, of the stationary vector process

$$X_{t-1,t} = \begin{pmatrix} r_{t-1,t} & \sigma^2_{t-1,t} \end{pmatrix}^\top .$$

The components of the matrices $\Gamma^X(l)$ may in turn be expressed in terms of those of the autocovariance matrices of the stationary vector process

$$Y_t = \begin{pmatrix} r_t & r_t^2 \end{pmatrix}^\top ,$$

since

$$X_{t-1,t} = \sum_{j=1}^{1/\Delta} Y_{t-1+j\Delta}.$$

Finally, knowledge of the mean vector and the autocovariance matrices of the process Y_t is sufficient for analyzing the risk-return tradeoff implicit in equation (20). See Appendix C and D for the calculations.

3.3 Variance Premium

In this paper, we will consider two different definitions of the variance-risk premia. The first one is given by

$$vp_t^{(1)} \equiv E_t^{\mathbb{Q}}\left[\sigma_{r,t+1}^2\right] - E_t\left[\sigma_{r,t+1}^2\right] \quad \text{where} \quad \sigma_{r,t}^2 \equiv Var_t\left[r_{t,t+1}\right]. \tag{22}$$

This definition, considered in the theoretical framework of Bollerslev et al. (2009), does not have a model-free counterpart. Consequently, these authors made a couple of changes in their empirical application. In order to measure the first term in Eq. (22), they used the (square of the) VIX measure which is the conditional expectation of the integrated variance under the $\mathbb{Q}$-measure when one assumes a continuous time framework (see for instance Bollerslev et al.; 2011). When there is no drift, the conditional expectation of the integrated variance equals the conditional expectation of the realized variance computed as the sum of the squared returns whatever the discretization sampling (Meddahi; 2002 and Andersen et al.; 2004). This leads Bollerslev et al. (2009) to measure the second part of Eq. (22) as the expected value of the realized variance of a future period. This is why we adopt a second definition of the variance risk-premia given by

$$vp_t^{(2)} \equiv E_t^{\mathbb{Q}}\left[\sigma_{t,t+1}^2\right] - E_t\left[\sigma_{t,t+1}^2\right] \quad \text{where} \quad \sigma_{t,t+1}^2 \equiv \sum_{j=1}^{1/\Delta} r_{t+j\Delta}^2. \tag{23}$$

Proposition 3.1 *One has*

$$vp_t^{(1)} = \lambda_{vp}^{(1)\top}\zeta_t \quad \text{with} \quad \lambda_{vp}^{(1)} = \Upsilon_{1/\Delta}^{\mathbb{Q}} - \Upsilon_{1/\Delta}, \tag{24}$$

where the vectors $\Upsilon_{1/\Delta}^{\mathbb{Q}}$ *and* $\Upsilon_{1/\Delta}$ *are given in Eq. (E.11) of the Appendix.*

Likewise, one has

$$vp_t^{(2)} = \lambda_{vp}^{(2)\top}\zeta_t \quad \text{with} \quad \lambda_{vp}^{(2)} = \lambda_{vix} - \lambda_{rv}, \tag{25}$$

and

$$\lambda_{vix} = \left(\sum_{j=1}^{1/\Delta} \Psi_{j-1}^{\mathbb{Q}(2)}\right)^\top \quad \text{and} \quad \lambda_{rv} = \left(\sum_{j=1}^{1/\Delta} P^{j-1}\right)^\top \Psi_0^{(2)},$$

where the vectors $\Psi_{j-1}^{\mathbb{Q}(2)}$ *for* $j = 1, ..., 1/\Delta$, *and* $\Psi_0^{(2)}$ *are given in Eq. (E.9) and Eq. (D.6) respectively.*

One can now easily characterize the ability of the two measures of variance risk premia to predict future returns given that one can write, for $i = 1$ or 2,

$$\frac{r_{t,t+l}}{l} = \alpha_{kl}^{(i)} + \beta_{1,kl}^{(i)} vp_{t-k}^{(i)} + \epsilon_{t,t+l}^{(i,k)}, \tag{26}$$

for which population values of the intercept $\alpha_{kl}^{(i)}$ the slope $\beta_{kl}^{(i)}$ and the coefficient of determination $R_{i,kl}^2$ are given by

$$\alpha_{kl}^{(i)} = \frac{E\left[r_{t,t+l}\right]}{l} - \beta_{1,kl}^{(i)} E\left[vp_{t-k}^{(i)}\right], \quad \beta_{kl}^{(i)} = \frac{1}{l}\frac{Cov(vp_{t-k}^{(i)}, r_{t,t+l})}{Var[vp_{t-k}^{(i)}]} \text{ and } R_{i,kl}^2 = \frac{\left(Cov(vp_{t-k}^{(i)}, r_{t,t+l})\right)^2}{Var[vp_{t-k}^{(i)}]Var\left[r_{t,t+l}\right]}.$$

One can show that

$$\begin{aligned} E\left[vp_{t-k}^{(i)}\right] &= \lambda_{vp}^{(i)\top}\mu^\zeta \quad \text{and} \quad Var\left[vp_{t-k}^{(i)}\right] = \lambda_{vp}^{(i)\top}\Sigma^\zeta\lambda_{vp}^{(i)} \\ Cov\left(vp_{t-k}^{(i)}, r_{t,t+l}\right) &= \sum_{j=1}^{l/\Delta}\left(\Psi_0^{(1)}\right)^\top P^{k/\Delta+j-1}\Sigma^\zeta\lambda_{vp}^{(i)}. \end{aligned} \tag{27}$$

4 Empirical Stylized Facts

In Bonomo et al. (2011), we introduced a similar model at the monthly frequency to match a number of asset pricing moments and predictability statistics. Even though we have enriched the volatility process we still want to reproduce these stylized facts, so that we keep the long-run risk features of the model. The values for the period 1930-2012 are reported in Table 4. The values for the moments of consumption growth are the usual ones with a mean and volatility of around 2 percent. The volatility of dividend growth is of course higher at around 13 percent while the mean is around 1 percent. We observe a correlation of 0.50 between the two growth rate series. The mean of the log equity premium is close to 5 percent, while the risk-free rate mean is close to 1 percent and its volatility around 4 percent. The volatility of excess returns is around 20 percent. In terms of predictability of future returns by the price-dividend ratio, the R^2 is increasing from 3.5 percent at one year to 23 percent at 5 years.

Of course the main goal of this paper is to reproduce risk-return trade-off statistics at both short and long horizons. Daily stylized facts are captured in Figure 1. Panel A1 depicts the daily autocorrelations of up to a lag length of 90 days of three volatility measures. The VIX^2 represents the option-embedded expectation of the cumulative variation of the S&P 500 index over the next month plus a potential variance premium for bearing the corresponding volatility risk. The RV line captures the daily autocorrelations of the realized volatility over the next month. The realized volatility is computed either as the sum of the daily squared returns over the next 22 days ($SQFor$) or the sum of the daily realized volatilities over the next 22 days ($RVFor$). Finally, the variance premium (VP) is obtained by projecting the realized volatility on variables known at time t and subtracting the predicted value from the VIX^2. This is to capture the difference between the risk-neutral and the objective expectations of the forward integrated variance. The daily VIX^2 is the most persistent, while the variance premium shows a faster decay especially the one obtained with $SQFor$ measure of RV. Panel A2 exhibits the cross-correlations of the VIX^2 and the VP series with daily returns at up to 20 leads and lags. In the left part of the graphs in Panel A2 we observe mainly negative correlations between lagged returns and current measures of volatility. This effect dubbed the leverage effect following Black and Cox (1976) is well documented in the empirical literature on volatility (see detailed references in Bollerslev et al. (2011)). The mostly positive cross-correlations in the right part of the figure, which indicate the relation between current volatility and future returns, capture what has been referred to as the volatility feedback effect.

The mean and standard deviation of the three volatility series (VIX^2, RV and VP) are reported in Table 1 at both the daily and monthly frequencies. For the short sample excluding the financial crisis, 1990 to 2007, the mean and standard deviation of the variance premium are respectively 11 and 15 percent. This is the result of a mean of 33 percent and a standard deviation of close to 24 percent for VIX^2 and a mean of 22 percent and a standard deviation of 23 percent for the realized volatility (RV). To compute the expectation of RV under the objective measure at the daily frequency we use the HAR model (see Corsi (2009)) to project the realized volatility on past information, as in Bollerslev et al. (2011). Not surprisingly then our values for the moments are quite close to theirs[4]. For the monthly moments, the expectation of the realized volatility is based on either on the lagged past value or on the projection on one, two or three lags of the realized volatility. The monthly values for the moments of the variance premium are a bit higher than their daily counterparts for both samples.

Another set of stylized facts relates the variance premium to future returns at a lower frequency than daily but still considered short-run. Table 2 reports monthly regression results from one to twelve months. The magnitudes of the predictability varies from the daily to the monthly frequency, form the pre-crisis to the post-crisis sample, and to the method used to compute the variance premium. However, in most cases we observe the same pattern. The R^2 peaks at three months and declines monotonically up to 12 months to become often negligible. Predictability is stronger at the monthly frequency than at the daily frequency. These patterns have been reported by Bollerslev et al. (2009).

Finally, we consider in Table 3 the long-run risk-return trade-off put forward by Bandi and Perron (2008). They show that the dependence between excess market returns and past market variance increases with the horizon and is strong in the long run, that is between 6 and 10 years. For their sample, from 1952 to 2006, they find R^2 of 26 percent for returns and variances computed over 6 years and up to 73 percent for 10 years. In the long sample we selected, from 1930 to 2012, we observe the same increasing pattern in R^2 but their values are much more modest. For 8, 9 and 10 years, the values are 4, 6 and 15 percent.

5 Model Calibration and Risk-Return Tradeoff Implications

The challenge is to reproduce the previous stylized facts at high and low frequencies with the same parameters for both preferences and fundamentals. First, we will explain our calibration and then we will assess the capacity of the model described in the previous sections to match the empirical facts.

[4]The difference comes from the fact that the series for the realized volatility is not exactly the same

5.1 Calibration

The model is calibrated at the daily frequency with $\Delta = 1/22$, and daily parameter values are derived from monthly values used in Bansal et al. (2012) and Bonomo et al. (2011). The unconditional mean of monthly consumption growth is $\mu_x^M = 0.15 \times 10^{-2}$. The corresponding daily value is $\mu_x = \mu_x^M \Delta$. The unconditional mean and standard deviation of monthly conditional variance of consumption growth are $\sqrt{\mu_\sigma^M} = 0.7305 \times 10^{-2}$ and $\sigma_\sigma^M = 0.6263 \times 10^{-4}$. The unconditional mean and standard deviation of daily logarithmic conditional variance of consumption growth are then set as $\mu_\sigma = \ln\left(\mu_\sigma^M \Delta\right) - \left(\sigma_M^2 \Delta\right)/2$ and $\sigma_\sigma = \ln\left(1 + \left(\left(\sigma_\sigma^M \sqrt{\Delta}\right) / \left(\mu_\sigma^M \Delta\right)\right)^2\right)$. The monthly persistence of the conditional variance of consumption growth is $\phi^M = 0.995$ in Bonomo et al. (2011); we use this value for the more persistent component of the daily log conditional variance $z_{1,t}$ and assume $\phi_{1z}^{1/\Delta} \in \{0.995, 0.999\}$. Our base case values for the persistence and the kurtosis of the second component $z_{2,t}$ are $\phi_{2z}^{1/\Delta} \in \{0.50, 0.60, 0.70\}$ and $k_{2z} = 10$, while we assume the base case value of 0.125 for the ratio s_σ / s_{2z}. The parameter $\beta_{\rho\sigma}$ controls the conditional correlation between consumption and dividend growth rates. In the first three scenarios we set it at 0.07, implying that this correlation increases with macroeconomic uncertainty, while in the last one it is set to zero, which means that the correlation is constant. The unconditional correlation is set to 0.4 in all four scenarios.

For preferences, we maintain the same parameters over all four scenarios except for the elasticity of intertemporal substitution whihc takes the value 1.5 in one scenario and 0.75 in the other three.

5.2 Asset Pricing and Risk-Return Tradeoff Model Implications

5.2.1 Asset Pricing Moments

We collect in Table 4 the moments associated with the fundamentals (consumption and dividend growth) as well as with the equity and risk-free rates of return for the four scenarios we described in the calibration section above. We can see that the consumption and dividend processes are closely matched by our calibrated Markov switching process. For asset prices, we consider a set of moments, namely the expected value and the standard deviation of the equity excess returns, the real risk-free rate, and the price-dividend ratio. The first scenario (S1) seems to fit most moments the closest except the volatility of the price-dividend ratio (0.27 instead of 0.45 in the data) and the excess equity return (8.62 instead of 5.35 in the data). The mean and standard deviation of the risk-free rate are particularly well matched. The second scenario (S2), which differs only by the value of the elasticity of intertemporal substitution, set at 0.75, and the persistence of the second component $\phi_{2z}^{1/\Delta}$ of volatility affects, as expected, mainly the risk-free rate moments. Scenario S3, which differs form scenario S1 only by the value of elasticity of intertemporal substitution (0.75 instead of 1.5) lowers the equity premium and doubles the standard deviation of the price-dividend ratio, putting these two moments more in line with the data. Scenario S4, where we increase $\phi_{2z}^{1/\Delta}$ to 0.6 from 0.5 in S3 and where we put the correlation between consumption and dividend equal to 1, produces statistics that are very similar to S3.

The variance premium moments are reported in Table 5. In Section 3.3, we have described two methods to compute the model equivalents of the VIX^2 and the realized volatility. The first one, referenced as $1st$ in the table, relies on computing the risk neutral and the objective expectations of the conditional variance of returns. The second approach, referenced as $2nd$ in the table, is based on the risk neutral and objective expectations of the monthly sum of daily returns. The first method produces moments that match rather well the values for the VIX^2 especially for scenarios 1 and 3. For the RV, the first moment is also well matched by this approach but the volatility is too high for all scenarios. This results in a reasonable value for the model-produced mean of variance premium but too low a value for the standard deviation. Scenario S4 produces the highest value for the latter (12.60) compared to the value of 20 estimated with the data. The second approach produced a much higher mean for the variance premium, which is more in line with the $RVFor$ empirical way to compute the expectation of the variance premium (20.47 in Table 1), and a very low standard deviation.

As pointed out in Section 3.3, the prediction of the monthly realized variance based on squared daily returns (SQFor) equals the prediction the monthly realized variance based on daily realized variances (RVFor) when there is no drift. However, Table 5 highlights substantial differences, which indicates that the drift plays a role when one considers long periods of time like a month.

5.2.2 Predictability

In this section we will look in turn to the short-run predictability of returns by the variance premium and to the medium and long-run predictability of returns by the dividend-price ratio and by the past cumulative variance.

Table 6 reproduces the regression results of future returns on the current variance premium. The four scenarios offer the same pattern for the coefficients of determination, that is a peak in the R^2 at the two- or three-month interval and a monotonic decrease up to 12 months. One important shortcoming of the model, in line with our low estimate of the standard deviation of the variance premium, is the large valued of the slope coefficients. However the values decrease monotonically as in the data.

In the bottom part of the Table 7 we see clearly that all four scenarios have no problem reproducing the magnitude and the dynamic pattern of predictability of future returns by the price-dividend ratio, especially for S1 and S2, which shows that the position of the elasticity of intertemporal substitution above or below one has no bearing on the strength of return predictability. This is in line with what was found in Bonomo et al. (2011).

The final predictability issue relates to the long-term existence of the risk-return trade-off. Results of the regression of cumulative returns over a number of months (from 12 to 120) over the cumulative realized volatility over the same number of months. Again the pattern in the data, increasing R^2 as the horizon lengthens, is well captured by the model, irrespective of the scenario chosen. Therefore we provide a model for rationalizing the empirical fact put forward by Bandi and Perron (2008). The risk-return trade-off is hard to find in the short-run but comes out clearly in the long run.

5.2.3 High-frequency Dynamics

We have left for the end perhaps the most challenging stylized facts, the autocorrelations of the daily measures of the risk-neutral expectation of the integrated variance and of the variance premium, as well as their daily cross-correlations with returns. This is the risk-return trade-off at the high -frequency level. Even though we write the model at a daily frequency, the model has been initially conceived as a long-run risk model. The only difference with the original model in Bonomo et al. (2011) has been the addition in the volatility process of consumption growth of a less persistent component. Figure 2 plots the model-implied autocorrelations in Panel A1 on the left hand side and the model-implied cross-correlations on the right hand side in Panel A2. The model equivalent of VIX^2 is more persistent than the variance premium, as in the data, but it shows a slightly faster decay than in the data. The variance premium autocorrelation starts near 1 and goes to 0.1 at 90 lags, very much like in the data. For the cross-correlations, the two variance measures are in the negative to the left, and therefore exhibit a leverage effect, while they jump back to the positive to the right and show a decreasing pattern while remaining in the positive. This is also consistent with the data for the variance risk premium, but not so much for the VIX^2, where the feedback effect is negligible.

6 Conclusion

We have assessed the ability of a long-run risk where preferences display generalized disappointment aversion (Routledge and Zin, 2009) to capture various stylized facts, high-frequency, short-run and long-run, about the risk-return trade-off in addition to the usual asset pricing moments and the return predictability by the dividend-price ratio. We have therefore written the model developed in Bonomo et al. (2011) at the daily frequency and derived closed-form formulas for all these stylized facts. For the dynamics of the consumption growth process we have maintained a random walk in consumption with a stochastic volatility that includes two mean-reverting components, one much more persistent than the other. Moreover we maintain the same calibration as in Bonomo et al. (2011) for the preference parameters and the fundamentals.

Overall, our results are very supportive of the model. We manage to match rather well all empirical facts, moments as well as predictability patterns, for both asset pricing and risk-return trade-off statistics at all horizons. A remaining challenge remains the variance of the variance risk premium, which is too low in our model. We could of course find a calibration that does better in that dimension but it will be at the expense of other stylized facts. We will leave this difficult task for future work.

Appendix

A Stochastic Discount Factor and Valuation Ratios

Appendix A provides the analytical formulae of financial variables implied by the model at the high frequency level (here daily level). They are proved in Bonomo et al. (2011). The Markov chain is stationary with ergodic distribution and second moments given by:

$$E[\zeta_t] = \mu^{\zeta} \in \mathbb{R}_+^N, \tag{A.1}$$
$$E\left[\zeta_t \zeta_t^{\top}\right] = Diag\left(\mu_1^{\zeta}, .., \mu_N^{\zeta}\right) \text{ and } \Sigma^{\zeta} = Var[\zeta_t] = Diag\left(\mu_1^{\zeta}, .., \mu_N^{\zeta}\right) - \mu^{\zeta}\left(\mu^{\zeta}\right)^{\top},$$

where $Diag(u_1, .., u_N)$ is the $N \times N$ diagonal matrix whose diagonal elements are u_1,..,u_N.

In order to compute expectations conditional to the Markov chain, we make use of the following results: Let X and Y be two normally distributed random variables with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and covariance σ_{XY}. Then, we have

$$E[\exp(uX + vY) I(X < x)]$$
$$= \exp\left(u\mu_X + v\mu_Y + \frac{1}{2}\left(u^2\sigma_X^2 + 2uv\sigma_{XY} + v^2\sigma_Y^2\right)\right)\Phi\left(\frac{x-\mu_X}{\sigma_X} - u\sigma_X - v\frac{\sigma_{XY}}{\sigma_X}\right), \tag{A.2}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

We then show that the stochastic discount factor $M_{t,t+\Delta}$ can also be written as

$$M_{t,t+\Delta} = \delta^*_{t,t+\Delta} \exp(-\gamma g_{c,t+\Delta})\left[1 + \ell I(g_{c,t+\Delta} < -g_{v,t+\Delta} + \ln\theta)\right] \tag{A.3}$$

where

$$\ln \delta^*_{t,t+\Delta} = \zeta_t^{\top} A \zeta_{t+\Delta} \quad \text{and} \quad g_{v,t+\Delta} = \zeta_t^{\top} B \zeta_{t+\Delta} \tag{A.4}$$

and where the components of matrices A and B are explicitly defined by

$$a_{ij} = \ln\delta + \left(\frac{1}{\psi} - \gamma\right) b_{ij} - \ln\left[1 + \ell\theta^{1-\gamma}\sum_{j=1}^{N} p_{ij}\Phi(q_{ij})\right] \tag{A.5}$$
$$b_{ij} = \ln\left(\frac{\lambda_{1v,j}}{\lambda_{1z,i}}\right) \quad \text{and} \quad q_{ij} = \frac{-b_{ij} + \ln\theta - \mu_{c,i}}{\sqrt{\omega_{c,i}}}.$$

Likewise, one has:

Proposition A.1 ***Characterization of Welfare Valuation Ratios.*** *Let*

$$\frac{\mathcal{R}_t\left(V_{t+\Delta}\right)}{C_t}=\lambda_{1z}^{\top}\zeta_t \text{ and } \frac{V_t}{C_t}=\lambda_{1v}^{\top}\zeta_t$$

respectively denote the ratio of the certainty equivalent of future lifetime utility to current consumption and the ratio of lifetime utility to consumption. The components of the vectors λ_{1z} and λ_{1v} are given by:

$$\lambda_{1z,i}=\exp\left(\mu_{c,i}+\frac{1-\gamma}{2}\omega_{c,i}\right)\left(\sum_{j=1}^{N}p_{ij}^{*}\lambda_{1v,j}^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \tag{A.6}$$

$$\lambda_{1v,i}=\left\{(1-\delta)+\delta\lambda_{1z,i}^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}} \text{ if } \psi\neq 1 \text{ and } \lambda_{1v,i}=\lambda_{1z,i}^{\delta} \text{ if } \psi=1, \tag{A.7}$$

where the matrix $P^{*\top}=\left[p_{ij}^{*}\right]_{1\leq i,j\leq N}$ is defined in A.11.

In the following, $\odot$ denotes the component-by-component multiplication operator.

Proposition A.2 ***Characterization of Asset Prices.*** *Let*

$$\frac{P_{d,t}}{D_t}=\lambda_{1d}^{\top}\zeta_t,\quad \frac{P_{c,t}}{C_t}=\lambda_{1c}^{\top}\zeta_t \quad\text{and}\quad R_{f,t+\Delta}=\frac{1}{\lambda_{1f}^{\top}\zeta_t}$$

respectively denote the price-dividend ratio, the price-consumption ratio and the risk-free rate. The components of the vectors λ_{1d}, λ_{1c}, and λ_{1f} are given by:

$$\lambda_{1d,i}=\delta\left(\frac{1}{\lambda_{1z,i}}\right)^{\frac{1}{\psi}-\gamma}\exp\left(\mu_{cd,i}+\frac{\omega_{cd,i}}{2}\right)\left(\lambda_{1v}^{\frac{1}{\psi}-\gamma}\right)^{\top}P^{**}\left(Id-\delta A^{**}\left(\mu_{cd}+\frac{\omega_{cd}}{2}\right)\right)^{-1}e_i \tag{A.8}$$

$$\lambda_{1c,i}=\delta\left(\frac{1}{\lambda_{1z,i}}\right)^{\frac{1}{\psi}-\gamma}\exp\left(\mu_{cc,i}+\frac{\omega_{cc,i}}{2}\right)\left(\lambda_{1v}^{\frac{1}{\psi}-\gamma}\right)^{\top}P^{*}\left(Id-\delta A^{*}\left(\mu_{cc}+\frac{\omega_{cc}}{2}\right)\right)^{-1}e_i \tag{A.9}$$

$$\lambda_{1f,i}=\delta\exp\left(-\gamma\mu_{c,i}+\frac{\gamma^2}{2}\omega_{c,i}\right)\sum_{j=1}^{N}\tilde{p}_{ij}^{*}\left(\frac{\lambda_{1v,j}}{\lambda_{1z,i}}\right)^{\frac{1}{\psi}-\gamma} \tag{A.10}$$

where the vectors $\mu_{cd}=-\gamma\mu_c+\mu_d$, $\omega_{cd}=\omega_c+\omega_d-2\gamma\rho\odot\sqrt{\omega_c}\odot\sqrt{\omega_d}$, $\mu_{cc}=(1-\gamma)\mu_c$, $\omega_{cc}=(1-\gamma)^2\omega_c$, and the matrices $P^{**\top}=\left[p_{ij}^{**}\right]_{1\leq i,j\leq N}$ and $\tilde{P}^{*\top}=\left[\tilde{p}_{ij}^{*}\right]_{1\leq i,j\leq N}$ as well as the matrix functions $A^{**}(u)$ and $A^{*}(u)$ are defined in (A.13), (A.14), (A.12) and (A.15), respectively. The *vector e_i denotes the $N \times 1$ vector with all components equal to zero but the ith component is equal to one.*

The components of the matrix $P^{*\top} = \left[p^*_{ij}\right]_{1\leq i,j\leq N}$ in (A.6) and (A.9), and the matrix function $A^*(u)$ also in (A.9) are defined by:

$$p^*_{ij} = p_{ij}\frac{1+\ell\Phi\left(q_{ij}-(1-\gamma)\sqrt{\omega_{c,i}}\right)}{1+\ell\theta^{1-\gamma}\sum\limits_{j=1}^{N}p_{ij}\Phi\left(q_{ij}\right)} \tag{A.11}$$

$$A^*(u) = Diag\left(\exp\left(\left(\frac{1}{\psi}-\gamma\right)b_{11}+u_1\right),...,\exp\left(\left(\frac{1}{\psi}-\gamma\right)b_{NN}+u_N\right)\right)P^*, \tag{A.12}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.

The matrix $P^{**\top} = \left[p^{**}_{ij}\right]_{1\leq i,j\leq N}$ in (A.8), and the matrix $\tilde{P}^{*\top} = \left[\tilde{p}^*_{ij}\right]_{1\leq i,j\leq N}$ in (A.10) have their components given by:

$$p^{**}_{ij} = p_{ij}\frac{1+\ell\Phi\left(q_{ij}-\left(\rho_i\sqrt{\omega_{d,i}}-\gamma\sqrt{\omega_{c,i}}\right)\right)}{1+\ell\theta^{1-\gamma}\sum\limits_{j=1}^{N}p_{ij}\Phi\left(q_{ij}\right)} \tag{A.13}$$

$$\tilde{p}^*_{ij} = p_{ij}\frac{1+\ell\Phi\left(q_{ij}+\gamma\sqrt{\omega_{c,i}}\right)}{1+\ell\theta^{1-\gamma}\sum\limits_{j=1}^{N}p_{ij}\Phi\left(q_{ij}\right)}. \tag{A.14}$$

The matrix function $A^{**}(u)$ in (A.8) is defined by:

$$A^{**}(u) = Diag\left(\exp\left(\left(\frac{1}{\psi}-\gamma\right)b_{11}+u_1\right),...,\exp\left(\left(\frac{1}{\psi}-\gamma\right)b_{NN}+u_N\right)\right)P^{**}. \tag{A.15}$$

B Time Aggregation: Model-Implied Low Frequency Moments

Appendix B provides the first and second moments of the vector $(g_{c,t,t+h},\ g_{d,t,t+h},\ z_{d,t,t+h},\ r_{f,t,t+h},\ r_{t,t+h})^\top$, that is, consumption growth, dividend growth, log price-dividend ratio, log risk-free return and the excess log equity-return at lower frequencies, like monthly, which are implied by the model defined at the high frequency (here daily).

Given the postulated dynamics of endowment and the implied-dynamics of asset prices at the frequency Δ (at which several economic variables may be unobservable), we are interested in the properties of these quantities at lower frequencies. We defined frequency h consumption growth, dividend growth, log price-dividend ratio, log risk-free return and excess log equity return over the log risk free return as follows:

$$\begin{aligned} g_{c,t,t+h} &= \ln\left(\frac{C_{t,t+h}}{C_{t-h,t}}\right), \quad g_{d,t,t+h} = \ln\left(\frac{D_{t,t+h}}{D_{t-h,t}}\right) \quad \text{and} \quad z_{d,t,t+h} = \ln\left(\frac{\bar{P}_{d,t,t+h}}{D_{t,t+h}}\right) \\ r_{f,t,t+h} &= \sum_{j=1}^{h/\Delta} r_{f,t+j\Delta} \quad \text{and} \quad r_{t,t+h} = \sum_{j=1}^{h/\Delta} r_{t+j\Delta} \end{aligned} \tag{B.1}$$

where $r_{f,t+\Delta} = \nu_{1f}^{\top}\zeta_t$ with $\nu_{1f} = \ln \lambda_{1f}$, and where

$$C_{t,t+h} = \sum_{i=1}^{h/\Delta} C_{t+i\Delta}, \quad D_{t,t+h} = \sum_{i=1}^{h/\Delta} D_{t+i\Delta} \quad \text{and} \quad \bar{P}_{d,t,t+h} = \frac{1}{h/\Delta}\sum_{i=1}^{h/\Delta} P_{d,t+i\Delta}. \tag{B.2}$$

We show that

$$\begin{aligned} g_{c,t,t+h} &\approx g_{c,t+\Delta} + \sum_{j=1}^{h/\Delta-1}\left(1-\frac{j}{h/\Delta}\right)\left(g_{c,t+\Delta+j\Delta} + g_{c,t+\Delta-j\Delta}\right) \\ g_{d,t,t+h} &\approx g_{d,t+\Delta} + \sum_{j=1}^{h/\Delta-1}\left(1-\frac{j}{h/\Delta}\right)\left(g_{d,t+\Delta+j\Delta} + g_{d,t+\Delta-j\Delta}\right) \\ z_{d,t,t+h} &\approx -\ln\left(h/\Delta\right) + \frac{1}{h/\Delta}\sum_{j=1}^{h/\Delta} z_{d,t+j\Delta} \quad \text{where} \quad z_{d,t} = \ln\left(\frac{P_{d,t}}{D_t}\right) \\ &= \sum_{j=1}^{h/\Delta} z_{d,h,t+j\Delta} \quad \text{where} \quad z_{d,h,t} = \frac{z_{d,t} - \ln\left(h/\Delta\right)}{h/\Delta}. \end{aligned} \tag{B.3}$$

It follows that first and second moments of the low frequency vector process

$$L_{t,t+h} = \begin{pmatrix} g_{c,t,t+h} & g_{d,t,t+h} & z_{d,t,t+h} & r_{f,t,t+h} & r_{t,t+h} \end{pmatrix}^{\top}$$

are completely determined by those of the high frequency vector process

$$H_t = \begin{pmatrix} g_{c,t} & g_{d,t} & z_{d,h,t} & r_{f,t} & r_t \end{pmatrix}^{\top}.$$

The mean and the autocovariance matrices of the vector process H_t are defined by $\mu^H = E\left[H_t\right] = \left(\mu_1^H,\ \mu_2^H,\ \mu_3^H,\ \mu_4^H,\ \mu_5^H\right)^{\top}$ and

$$\Gamma^H(j) = Cov(H_t, H_{t+j\Delta}) = \begin{bmatrix} \gamma_{11}^H(j) & \gamma_{12}^H(j) & \gamma_{13}^H(j) & \gamma_{14}^H(j) & \gamma_{15}^H(j) \\ \gamma_{21}^H(j) & \gamma_{22}^H(j) & \gamma_{23}^H(j) & \gamma_{24}^H(j) & \gamma_{25}^H(j) \\ \gamma_{31}^H(j) & \gamma_{32}^H(j) & \gamma_{33}^H(j) & \gamma_{34}^H(j) & \gamma_{35}^H(j) \\ \gamma_{41}^H(j) & \gamma_{42}^H(j) & \gamma_{43}^H(j) & \gamma_{44}^H(j) & \gamma_{45}^H(j) \\ \gamma_{51}^H(j) & \gamma_{52}^H(j) & \gamma_{53}^H(j) & \gamma_{54}^H(j) & \gamma_{55}^H(j) \end{bmatrix}. \tag{B.4}$$

We have

$$\begin{aligned} &\mu_1^H = \mu_c^\top \mu^\zeta, \quad \mu_2^H = \mu_d^\top \mu^\zeta \quad \text{and} \quad \mu_3^H = \nu_{1d,h}^\top \mu^\zeta, \quad \text{where} \quad \nu_{1d,h} = \frac{\nu_{1d} - \ln(h/\Delta)}{h/\Delta}, \\ &\mu_4^H = \nu_{1f}^\top \mu^\zeta \quad \text{and} \quad \mu_5^H = \mu_r^\top \mu^\zeta, \quad \text{with} \quad \nu_{1d} = \ln \lambda_{1d}, \end{aligned} \tag{B.5}$$

where μ_r is the diagonal of ΛP, and $\forall j \geq 0$ we have

$$\begin{aligned}
&\gamma_{11}^H(j) = \mu_c^\top P^j \Sigma^\zeta \mu_c + \left(\omega_c^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{12}^H(j) = \mu_d^\top P^j \Sigma^\zeta \mu_c + \left(\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{13}^H(j) = \nu_{1d,h}^\top P^{j+1} \Sigma^\zeta \mu_c \quad \text{and} \quad \gamma_{14}^H(j) = \nu_{1f}^\top P^j \Sigma^\zeta \mu_c \\
&\gamma_{15}^H(j) = \mu_r^\top P^j \Sigma^\zeta \mu_c + \left(\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{21}^H(j) = \mu_c^\top P^j \Sigma^\zeta \mu_d + \left(\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{22}^H(j) = \mu_d^\top P^j \Sigma^\zeta \mu_d + \left(\omega_d^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{23}^H(j) = \nu_{1d,h}^\top P^{j+1} \Sigma^\zeta \mu_d \quad \text{and} \quad \gamma_{24}^H(j) = \nu_{1f}^\top P^j \Sigma^\zeta \mu_d \\
&\gamma_{25}^H(j) = \mu_r^\top P^j \Sigma^\zeta \mu_d + \left(\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{31}^H(j) = \left(\nu_{1d,h}^\top P \Sigma^\zeta \mu_c\right) I(j=0) + \left(\mu_c^\top P^{j-1} \Sigma^\zeta \nu_{1d,h}\right) I(j \geq 1) \\
&\gamma_{32}^H(j) = \left(\nu_{1d,h}^\top P \Sigma^\zeta \mu_d\right) I(j=0) + \left(\mu_d^\top P^{j-1} \Sigma^\zeta \nu_{1d,h}\right) I(j \geq 1) \\
&\gamma_{33}^H(j) = \nu_{1d,h}^\top P^j \Sigma^\zeta \nu_{1d,h} \quad \text{and} \quad \gamma_{34}^H(j) = \left(\nu_{1d,h}^\top P \Sigma^\zeta \nu_{1f}\right) I(j=0) + \left(\nu_{1f}^\top P^{j-1} \Sigma^\zeta \nu_{1d,h}\right) I(j \geq 1) \\
&\gamma_{35}^H(j) = \left(\breve{\nu}_{1d,h,1}^\top \mu^\zeta - \left(\nu_{1d,h}^\top \mu^\zeta\right)\left(\mu_r^\top \mu^\zeta\right)\right) I(j=0) + \left(\mu_r^\top P^{j-1} \Sigma^\zeta \nu_{1d,h}\right) I(j \geq 1) \\
&\gamma_{41}^H(j) = \mu_c^\top P^j \Sigma^\zeta \nu_{1f}, \quad \gamma_{42}^H(j) = \mu_d^\top P^j \Sigma^\zeta \nu_{1f} \quad \text{and} \quad \gamma_{43}^H(j) = \nu_{1d,h}^\top P^{j+1} \Sigma^\zeta \nu_{1f}
\end{aligned} \tag{B.6}$$

$$\gamma_{44}^{H}(j) = \nu_{1f}^{\top} P^{j} \Sigma^{\zeta} \nu_{1f} \quad \text{and} \quad \gamma_{45}^{H}(j) = \mu_{r}^{\top} P^{j} \Sigma^{\zeta} \nu_{1f}$$

$$\gamma_{51}^{H}(j) = \left(\mu_{c}^{\top} \Sigma^{\zeta} \mu_{r} + \left(\rho \odot \sqrt{\omega_{c}} \odot \sqrt{\omega_{d}}\right)^{\top} \mu^{\zeta}\right) I(j=0) + \left(\breve{\mu}_{c,j}^{\top} \mu^{\zeta} - \left(\mu_{c}^{\top} \mu^{\zeta}\right)\left(\mu_{r}^{\top} \mu^{\zeta}\right)\right) I(j \geq 1)$$

$$\gamma_{52}^{H}(j) = \left(\mu_{d}^{\top} \Sigma^{\zeta} \mu_{r} + \omega_{d}^{\top} \mu^{\zeta}\right) I(j=0) + \left(\breve{\mu}_{d,j}^{\top} \mu^{\zeta} - \left(\mu_{d}^{\top} \mu^{\zeta}\right)\left(\mu_{r}^{\top} \mu^{\zeta}\right)\right) I(j \geq 1)$$

$$\gamma_{53}^{H}(j) = \breve{\nu}_{1d,h,j+1}^{\top} \mu^{\zeta} - \left(\nu_{1d,h}^{\top} \mu^{\zeta}\right)\left(\mu_{r}^{\top} \mu^{\zeta}\right)$$

$$\gamma_{54}^{H}(j) = \left(\nu_{1f}^{\top} \Sigma^{\zeta} \mu_{r}\right) I(j=0) + \left(\breve{\nu}_{1f,j}^{\top} \mu^{\zeta} - \left(\mu_{d}^{\top} \mu^{\zeta}\right)\left(\mu_{r}^{\top} \mu^{\zeta}\right)\right) I(j \geq 1)$$

$$\gamma_{55}^{H}(j) = \left(\left(\mu_{r}^{(2)} + \omega_{d}\right)^{\top} \mu^{\zeta} - \left(\mu_{r}^{\top} \mu^{\zeta}\right)^{2}\right) I(j=0) + \left(\breve{\mu}_{r,j}^{\top} \mu^{\zeta} - \left(\mu_{r}^{\top} \mu^{\zeta}\right)^{2}\right) I(j \geq 1)$$

with $\gamma_{nq}^{H}(-j) = \gamma_{qn}^{H}(j)$ for $n, q \in \{1,2,3,4,5\}$, where of a given vector u, we have $\breve{u}_j$ is the diagonal of the matrix $\left(\left(eu^{\top} P^{j-1}\right) \odot \Lambda\right) P$ and where $\mu_r^{(2)}$ is the diagonal of the matrix $(\Lambda \odot \Lambda) P$.

We show that the components of the mean of the vector process $L_{t,t+h}$ are given by:

$$\mu_{i}^{L} = (h/\Delta)\, \mu_{i}^{H}, \quad \text{for} \quad i \in \{1,2,3,4,5\}. \tag{B.7}$$

The 5×5 autocovariance matrices of the vector process $L_{t,t+h}$ are defined by

$$\Gamma^{L}(k) = Cov\left(L_{t,t+h}, L_{t+kh,t+(k+1)h}\right). \tag{B.8}$$

We have:

$$\begin{aligned}
\gamma_{nq}^{L}(k) &= \gamma_{nq}^{H}\left(\frac{kh}{\Delta}\right) + 2 \sum_{j=1}^{h/\Delta-1} \left(1 - \frac{j}{h/\Delta}\right)\left(\gamma_{nq}^{H}\left(\frac{kh}{\Delta} + j\right) + \gamma_{nq}^{H}\left(\frac{kh}{\Delta} - j\right)\right) \\
&\quad + \sum_{j=1}^{h/\Delta-1} \sum_{i=1}^{h/\Delta-1} \left(1 - \frac{j}{h/\Delta}\right)\left(1 - \frac{i}{h/\Delta}\right)\left(\gamma_{nq}^{H}\left(\frac{kh}{\Delta} + i - j\right) + \gamma_{nq}^{H}\left(\frac{kh}{\Delta} - i - j\right)\right. \\
&\qquad \left. \gamma_{nq}^{H}\left(\frac{kh}{\Delta} + i + j\right) + \gamma_{nq}^{H}\left(\frac{kh}{\Delta} - i + j\right)\right) \\
\gamma_{nl}^{L}(k) &= \sum_{i=1}^{h/\Delta} \gamma_{nl}^{H}\left(\frac{kh}{\Delta} + i - 1\right) \\
&\quad + \sum_{j=1}^{h/\Delta-1} \sum_{i=1}^{h/\Delta} \left(1 - \frac{j}{h/\Delta}\right)\left(\gamma_{nl}^{H}\left(\frac{kh}{\Delta} + i - j - 1\right) + \gamma_{nl}^{H}\left(\frac{kh}{\Delta} + i + j - 1\right)\right) \\
\gamma_{lq}^{L}(k) &= \sum_{i=1}^{h/\Delta} \gamma_{lq}^{H}\left(\frac{kh}{\Delta} - i + 1\right)
\end{aligned} \tag{B.9}$$

$$+\sum_{j=1}^{h/\Delta-1}\sum_{i=1}^{h/\Delta}\left(1-\frac{j}{h/\Delta}\right)\left(\gamma_{lq}^{H}\left(\frac{kh}{\Delta}-i+j+1\right)+\gamma_{lq}^{H}\left(\frac{kh}{\Delta}-i-j+1\right)\right)$$

for all $n, q \in \{1,2\}$ and $l \in \{3,4,5\}$.

We also have

$$\gamma_{nq}^{L}(k)=\frac{h}{\Delta}\gamma_{nq}^{H}\left(\frac{kh}{\Delta}\right)+\sum_{j=1}^{h/\Delta-1}\left(\frac{h}{\Delta}-j\right)\left(\gamma_{nq}^{H}\left(\frac{kh}{\Delta}+j\right)+\gamma_{nq}^{H}\left(\frac{kh}{\Delta}-j\right)\right) \tag{B.10}$$

for all $n, q \in \{3,4,5\}$.

C Risk-Return Trade-off

The autocovariance matrices of the vector process $X_{t-1,t}$ are defined by

$$\Gamma^{X}(l)=Cov\left(X_{t-1,t},X_{t+l-1,t+l}\right)=\begin{bmatrix}\gamma_{11}^{X}(l) & \gamma_{12}^{X}(l)\\ \gamma_{21}^{X}(l) & \gamma_{22}^{X}(l)\end{bmatrix}. \tag{C.1}$$

The variances of long-horizon returns and long-horizon realized variance, as well as their covariances, can be expressed as follows:

$$Var\left[\begin{pmatrix} r_{t,t+h}\\ \sigma_{t,t+h}^{2}\end{pmatrix}\right]=Var\left[\begin{pmatrix} r_{t-h,t}\\ \sigma_{t-h,t}^{2}\end{pmatrix}\right]=h\Gamma^{X}(0)+\sum_{l=1}^{h-1}(h-l)\left(\Gamma^{X}(l)+\Gamma^{X}(l)^{\top}\right). \tag{C.2}$$

The covariance of future long-horizon returns with past long-horizon realized variance can be expressed as follows:

$$Cov\left(\sigma_{t-m,t}^{2},r_{t,t+h}\right)=\min(m,h)\sum_{l=\min(m,h)}^{\max(m,h)}\gamma_{21}^{X}(l)+\sum_{l=1}^{\min(m,h)-1}l\left(\gamma_{21}^{X}(l)+\gamma_{21}^{X}(m+h-l)\right). \tag{C.3}$$

The covariance of past long-horizon returns with future long-horizon realized variance can be expressed as follows:

$$Cov\left(r_{t-m,t},\sigma_{t,t+h}^{2}\right)=\min(m,h)\sum_{l=\min(m,h)}^{\max(m,h)}\gamma_{12}^{X}(l)+\sum_{l=1}^{\min(m,h)-1}l\left(\gamma_{12}^{X}(l)+\gamma_{12}^{X}(m+h-l)\right). \tag{C.4}$$

We also have that $\forall l$ and $\forall n, q \in \{1,2\}$,

$$\gamma_{nq}^{X}(l) = \frac{1}{\Delta}\gamma_{nq}^{Y}\left(\frac{l}{\Delta}\right) + \sum_{j=1}^{1/\Delta-1}\left(\frac{1}{\Delta}-j\right)\left(\gamma_{nq}^{Y}\left(\frac{l}{\Delta}+j\right)+\gamma_{nq}^{Y}\left(\frac{l}{\Delta}-j\right)\right). \tag{C.5}$$

D Leverage and Volatility Feedback Effects

The autocovariance matrices of the vector process Y_t are defined by

$$\Gamma^{Y}(j) = Cov\left(Y_t, Y_{t+j\Delta}\right) = \begin{bmatrix} \gamma_{11}^{Y}(j) & \gamma_{12}^{Y}(j) \\ \gamma_{21}^{Y}(j) & \gamma_{22}^{Y}(j) \end{bmatrix}. \tag{D.1}$$

We recall the property $\forall j \geq 0,\ E_t\left[\zeta_{t+j\Delta}\right] = P^j \zeta_t$. Let $Y_t^{(n)}$ denotes the nth component of the vector process Y_t, for example $Y_t^{(2)} \equiv r_t^2$.

We now adopt the following notations, $\forall n, q \in \{1,2\}$:

$$\begin{aligned} E_t\left[Y_{t+\Delta}^{(n)} \mid \zeta_{k\Delta}, k \in \mathbb{Z}\right] &= \zeta_t^{\top} U^{(n)} \zeta_{t+\Delta}, \\ E_t\left[Y_{t+\Delta}^{(n)} Y_{t+\Delta}^{(q)} \mid \zeta_{k\Delta}, k \in \mathbb{Z}\right] &= \zeta_t^{\top} U^{(nq)} \zeta_{t+\Delta}. \end{aligned} \tag{D.2}$$

We show that:

$$U^{(1)} = \Lambda \quad \text{and} \quad U^{(2)} = (\Lambda \odot \Lambda) + \omega_d e^{\top}. \tag{D.3}$$

We also show that:

$$\begin{aligned} U^{(11)} &= (\Lambda \odot \Lambda) + \omega_d e^{\top} \\ U^{(12)} = U^{(21)} &= (\Lambda \odot \Lambda \odot \Lambda) + 3\Lambda \odot \left(\omega_d e^{\top}\right) \\ U^{(22)} &= (\Lambda \odot \Lambda \odot \Lambda \odot \Lambda) + 6\left(\Lambda \odot \Lambda\right) \odot \left(\omega_d e^{\top}\right) + 3\left(\omega_d \odot \omega_d\right) e^{\top}. \end{aligned} \tag{D.4}$$

We also adopt the following notations, $\forall n, q \in \{1,2\}$:

$$\begin{aligned} E_t\left[Y_{t+\Delta+j\Delta}^{(n)}\right] &= \left(\Psi_0^{(n)}\right)^{\top} P^j \zeta_t, \\ E_t\left[Y_{t+\Delta}^{(n)} Y_{t+\Delta+j\Delta}^{(q)}\right] &= \left(\Psi_j^{(nq)}\right)^{\top} \zeta_t, \quad \forall j \geq 0. \end{aligned} \tag{D.5}$$

We show that, $\forall n, q \in \{1,2\}$:

$$
\begin{aligned}
&\Psi_0^{(n)} \text{ is the diagonal of the matrix } U^{(n)}P, \\
&\Psi_0^{(nq)} \text{ is the diagonal of the matrix } U^{(nq)}P, \\
&\Psi_j^{(nq)} \text{ is the diagonal of the matrix } \left(U^{(n)} \odot \left(e\left(\Psi_0^{(q)}\right)^\top P^{j-1}\right)\right)P, \quad \forall j \geq 1.
\end{aligned}
\tag{D.6}
$$

Finally we have that, $\forall n, q \in \{1,2\}$:

$$
\begin{aligned}
\mu_n^Y &= E\left[Y_t^{(n)}\right] = \left(\Psi_0^{(n)}\right)^\top \mu^\zeta, \\
\gamma_{nq}^Y(j) &= \left(\left(\Psi_j^{(nq)}\right)^\top \mu^\zeta\right) - \left(\left(\Psi_0^{(n)}\right)^\top \mu^\zeta\right)\left(\left(\Psi_0^{(q)}\right)^\top \mu^\zeta\right), \quad \forall j \geq 0.
\end{aligned}
\tag{D.7}
$$

E Variance Premium

E.1 Dynamics under the $\mathbb{Q}$-measure

Henceforth, dynamics under the risk-neutral ($\mathbb{Q}$) measure will be represented with $\mathbb{Q}$ subscript.

Dynamics of the Markov-chain: We have

$$
\begin{aligned}
E_t^{\mathbb{Q}}\left[\zeta_{t+\Delta}\right] &= E_t\left[M_{t,t+\Delta} R_{f,t+\Delta} \zeta_{t+\Delta}\right] \\
&= \ldots \\
&= E_t\left[\zeta_{t+\Delta}\zeta_{t+\Delta}^\top\right]\left(\tilde{M} \odot \left(\lambda_{2f} e^\top\right)\right)^\top \zeta_t \\
&= \left(Diag\left(e_1^\top P\zeta_t, \ldots, e_N^\top P\zeta_t\right)\right)\left(\tilde{M} \odot \left(\lambda_{2f} e^\top\right)\right)^\top \zeta_t \\
&= \ldots \\
&= \mathcal{E}\left(\left(\tilde{M} \odot \left(\lambda_{2f} e^\top\right)\right)^\top \otimes P\right)\mathcal{E}^\top \zeta_t
\end{aligned}
\tag{E.1}
$$

where $\mathcal{E}$ is the $N \times N^2$ matrix such that the ith row is the vector $(e_i \otimes e_i)^\top$, where the components of the matrix $\tilde{M}$ are given by:

$$
\tilde{m}_{ij} = \exp\left(a_{ij} - \gamma\mu_{c,i} + \frac{1}{2}\gamma^2\omega_{c,i}\right)\left[1 + \ell\Phi\left(q_{ij} + \gamma\sqrt{\omega_{c,i}}\right)\right], \tag{E.2}
$$

and where $\lambda_{2f} = 1/\lambda_{1f}$. It follows that, under the risk-neutral measure, the Markov chain s_t has the one-period transition probability matrix

$$P^{\mathbb{Q}} = \mathcal{E}\left(\left(\tilde{M} \odot \left(\lambda_{2f} e^\top\right)\right)^\top \otimes P\right)\mathcal{E}^\top.$$

Let $P^{\mathbb{Q}(j)}$ be the j-period transition probability matrix under the risk neutral measure, defined by

$$E_t^{\mathbb{Q}}\left[\zeta_{t+j\Delta}\right] = P^{\mathbb{Q}(j)}\zeta_t. \tag{E.3}$$

We show that $P^{\mathbb{Q}(j)},\ j \geq 1$ satisfies the recursion

$$P^{\mathbb{Q}(j)} = P^{\mathbb{Q}(j-1)}\mathcal{E}\left(\left(\lambda_{1f}^{(j-1)}\left(\lambda_{1f}^{(1)} \odot \lambda_{2f}^{(j)}\right)^\top\right) \otimes P^{\mathbb{Q}}\right)\mathcal{E}^\top \quad \text{with} \quad P^{\mathbb{Q}(1)} = P^{\mathbb{Q}},$$

where $\lambda_{1f}^{(j)}$ is the vector of j-period risk-free bond prices, defined by

$$E_t\left[M_{t,t+j\Delta}\right] = \lambda_{1f}^{(j)\top}\zeta_t, \tag{E.4}$$

and satisfying the recursion

$$\lambda_{1f}^{(j)} = \lambda_{1f} \odot \left(P^{\mathbb{Q}\top}\lambda_{1f}^{(j-1)}\right) \quad \text{with} \quad \lambda_{1f}^{(1)} = \lambda_{1f}.$$

Dynamics of the returns and squared returns: We adopt the following notation, $\forall n \in \{1, 2\}$:

$$E_t\left[M_{t,t+\Delta}Y_{t+\Delta}^{(n)} \mid \zeta_{k\Delta}, k \in \mathbb{Z}\right] = \zeta_t^\top U^{\mathbb{Q}(n)}\zeta_{t+\Delta}. \tag{E.5}$$

We show that

$$\begin{aligned} U^{\mathbb{Q}(1)} = \exp\left(A - \gamma\mu_c e^\top + \frac{\gamma^2}{2}\omega_c e^\top\right) \odot \Big[&\left(\Lambda - \gamma\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)e^\top\right) \odot \left(1 + \ell\Phi\left(Q + \gamma\sqrt{\omega_c}e^\top\right)\right) \\ &-\ell\left(\left(\rho \odot \sqrt{\omega_d}\right)e^\top\right) \odot \phi\left(Q + \gamma\sqrt{\omega_c}e^\top\right)\Big] \end{aligned} \tag{E.6}$$

$$
\begin{aligned}
U^{\mathbb{Q}(2)} = \exp\left(A - \gamma\mu_c e^\top + \frac{\gamma^2}{2}\omega_c e^\top\right) \odot \\
&\Big[\Big(\omega_d e^\top + \Big(\Lambda - \gamma\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)e^\top\Big) \odot \Big(\Lambda - \gamma\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)e^\top\Big)\Big) \odot \\
&\quad \Big(1 + \ell\Phi\left(Q + \gamma\sqrt{\omega_c}e^\top\right)\Big) \\
&- 2\ell\left(\left(\rho \odot \sqrt{\omega_d}\right)e^\top\right) \odot \Big(\Lambda - \gamma\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)e^\top\Big) \odot \phi\left(Q + \gamma\sqrt{\omega_c}e^\top\right) \\
&-\ell\left(\left(\rho \odot \sqrt{\omega_d}\right)e^\top\right) \odot \left(\left(\rho \odot \sqrt{\omega_d}\right)e^\top\right) \odot \left(Q + \gamma\sqrt{\omega_c}e^\top\right) \odot \phi\left(Q + \gamma\sqrt{\omega_c}e^\top\right)\Big].
\end{aligned}
\tag{E.7}
$$

We also adopt the following notation, $\forall n \in \{1, 2\}$:

$$
E_t^{\mathbb{Q}}\left[Y_{t+\Delta+j\Delta}^{(n)}\right] = \left(\Psi_j^{\mathbb{Q}(n)}\right)^\top \zeta_t, \quad \forall j \geq 0. \tag{E.8}
$$

We show that $\Psi_j^{\mathbb{Q}(n)},\ j \geq 0$ satisfies the recursion

$$
\Psi_j^{\mathbb{Q}(n)} = \left(\lambda_{1f}^{(1)} \odot \lambda_{2f}^{(j+1)}\right) \odot \left(P^{\mathbb{Q}\top}\left(\lambda_{1f}^{(j)} \odot \Psi_{j-1}^{\mathbb{Q}(n)}\right)\right) \tag{E.9}
$$

with the initial condition

$$
\Psi_0^{\mathbb{Q}(n)} \text{ is the diagonal of the matrix } \left(\lambda_{2f}e^\top\right) \odot \left(U^{\mathbb{Q}(n)}P\right). \tag{E.10}
$$

E.2 Proof of Proposition 3.1

The Markov property of the model implies that

$$
\sigma_{r,t}^2 \equiv Var_t\left[r_{t,t+1}\right] = \omega_r^\top \zeta_t.
$$

We have:

$$
\begin{aligned}
\omega_r^\top \zeta_t = Var_t\left[r_{t,t+1}\right] = Var_t\left[\sum_{j=1}^{1/\Delta} r_{t+j\Delta}\right] \\
&= \sum_{j=1}^{1/\Delta} Var_t\left[r_{t+j\Delta}\right] + 2\sum_{j=2}^{1/\Delta}\sum_{i=1}^{j-1} Cov_t\left(r_{t+i\Delta}, r_{t+j\Delta}\right).
\end{aligned}
$$

Based on previous calculations, we have:

$$
\begin{aligned}
Var_t\left[r_{t+j\Delta}\right] &= \left(\Psi_0^{(2)}\right)^\top P^{j-1}\zeta_t - \left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\zeta_t\right)^2 \\
Cov_t\left(r_{t+i\Delta}, r_{t+j\Delta}\right) &= \left(\Psi_{j-i}^{(11)}\right)^\top P^{i-1}\zeta_t - \left(\left(\Psi_0^{(1)}\right)^\top P^{i-1}\zeta_t\right)\left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\zeta_t\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\omega_r = &\sum_{j=1}^{1/\Delta}\left(\left(\Psi_0^{(2)}\right)^\top P^{j-1} - \left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\right)\odot\left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\right)\right)^\top \\
&+ 2\sum_{j=2}^{1/\Delta}\sum_{i=1}^{j-1}\left(\left(\Psi_{j-i}^{(11)}\right)^\top P^{i-1} - \left(\left(\Psi_0^{(1)}\right)^\top P^{i-1}\right)\odot\left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\right)\right)^\top.
\end{aligned}
$$

The Markov property of the model implies

$$
E_t\left[\sigma_{r,t+j\Delta}^2\right] = \Upsilon_j^\top\zeta_t \quad \text{and} \quad E_t^{\mathbb{Q}}\left[\sigma_{r,t+j\Delta}^2\right] = \Upsilon_j^{\mathbb{Q}\top}\zeta_t,
$$

and we show that:

$$
\Upsilon_j = \left(\omega_r^\top P^j\right)^\top \quad \text{and} \quad \Upsilon_j^{\mathbb{Q}} = \left(\omega_r^\top P^{\mathbb{Q}(j)}\right)^\top. \tag{E.11}
$$

It follows that

$$
E_t\left[\sigma_{r,t+1}^2\right] = \Upsilon_{1/\Delta}^\top\zeta_t \quad \text{and} \quad E_t^{\mathbb{Q}}\left[\sigma_{r,t+1}^2\right] = \Upsilon_{1/\Delta}^{\mathbb{Q}\top}\zeta_t,
$$

which implies Eq. (24).

One can show that

$$
E_t\left[\sigma_{t,t+1}^2\right] = \left(\Psi_0^{(2)}\right)^\top\left(\sum_{j=1}^{1/\Delta}P^{j-1}\right)\zeta_t \quad \text{and} \quad E_t^{\mathbb{Q}}\left[\sigma_{t,t+1}^2\right] = \sum_{j=1}^{1/\Delta}\left(\Psi_{j-1}^{\mathbb{Q}(2)}\right)^\top\zeta_t,
$$

which leads to Eq. (25)

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Table 1: Variance Premium Moments

The entries of the table are the first and second moments of the variance premium, the options implied variance and the realized variance. In computing the daily variance premium, expected realized variance is a statistical forecast of realized variance using the Heterogeneous Autoregressive model of Realized Variance (HAR-RV). The realized variance is the sum of squared 5-minute (RVFor) or the sum of squared daily log returns of the S&P 500 index over a 22-day period and its risk-neutral expectation is measured as the end-of-period VIX-squared de-annualized ($VIX^2/12$). In computing the monthly variance premium, expected realized variance is simply the lag realized variance or a statistical forecast of realized variance using and $AR(p)$ model. The realized variance is the sum of squared 5-minute log returns of the S&P 500 index over the period and its risk-neutral expectation is measured as the end-of-period VIX-squared de-annualized ($VIX^2/12$). All measures are of monthly basis in percentage-squared.

Full Sample: January 1990 to December 2012							
Daily data			Monthly data				
Moments	RVFor	SQFor	Moments	Lag	$AR(1)$	$AR(2)$	$AR(3)$
$E[VRP]$	20.47	9.73	$E[VRP]$	18.41	18.40	18.40	18.40
$\sigma[VRP]$	22.08	20.19	$\sigma[VRP]$	20.40	19.89	26.87	31.36
$AC1(VRP)$	0.864	0.768	$AC1(VRP)$	0.254	0.555	0.620	0.615
$E[VIX^2]$	39.84		$E[VIX^2]$	39.79			
$\sigma[VIX^2]$	40.23		$\sigma[VIX^2]$	35.72			
$AC1(VIX^2)$	0.971		$AC1(VIX^2)$	0.804			
$E[RV]$	19.37	30.11	$E[RV]$	21.39			
$\sigma[RV]$	33.76	52.53	$\sigma[RV]$	37.60			
$AC1(RV)$	0.997	0.994	$AC1(RV)$	0.649			

Subsample: January 1990 to October 2007							
Daily data			Monthly data				
Moments	RVFor	SQFor	Moments	Lag	$AR(1)$	$AR(2)$	$AR(3)$
$E[VRP]$	18.78	11.16	$E[VRP]$	20.67	20.97	21.20	21.49
$\sigma[VRP]$	16.23	15.13	$\sigma[VRP]$	16.03	17.59	20.73	21.87
$AC1(VRP)$	0.938	0.910	$AC1(VRP)$	0.419	0.569	0.559	0.664
$E[VIX^2]$	32.76		$E[VIX^2]$	36.78			
$\sigma[VIX^2]$	23.75		$\sigma[VIX^2]$	25.27			
$AC1(VIX^2)$	0.976		$AC1(VIX^2)$	0.756			
$E[RV]$	13.98	21.61	$E[RV]$	16.23			
$\sigma[RV]$	14.15	23.12	$\sigma[RV]$	17.04			
$AC1(RV)$	0.997	0.990	$AC1(RV)$	0.709			

Table 2: Short-Run Risk-Return Trade-Offs

The entries of the table are the slope coefficients as well as the coefficient of determination (R_l^2) of the regression

$$\frac{r_{t,t+l}}{l} = \alpha_{0l} + \beta_{1,0l} vp_t + \epsilon^{(0)}_{t,t+l}$$

where vp_t is the current variance premium and $r_{t,t+l}$ is the accumulated future returns over l months. In computing the daily variance premium, expected realized variance is a statistical forecast of realized variance using the Heterogeneous Autoregressive model of Realized Variance (HAR-RV). The realized variance is the sum of squared 5-minute (RVFor) or the sum of squared daily (SQFor) log returns of the S&P 500 index over a 22-day period and its risk-neutral expectation is measured as the end-of-period VIX-squared de-annualized ($VIX^2/12$). In computing the monthly variance premium, expected realized variance is simply the lag realized variance or a statistical forecast of realized variance using and $AR(p)$ model. The realized variance is the sum of squared 5-minute log returns of the S&P 500 index over the period and its risk-neutral expectation is measured as the end-of-period VIX-squared de-annualized ($VIX^2/12$). All measures are of monthly basis in percentage-squared.

l	1	3	6	9	12	l	1	3	6	9	12
Daily data						Monthly data					
Full Sample: January 1990 to December 2012											
Forecast of RV based on RVFor						Forecast of RV is simply lag RV					
$\hat{\beta}_{1,0l}$	2.70	1.95	1.90	1.30	1.05	$\hat{\beta}_{1,0l}$	5.14	4.49	2.87	1.72	1.32
$se\left(\hat{\beta}_{1,0l}\right)$	1.26	1.24	0.68	0.56	0.53	$se\left(\hat{\beta}_{1,0l}\right)$	1.27	0.76	0.68	0.60	0.52
R_l^2	1.54	2.49	4.28	2.94	2.44	R_l^2	5.13	11.66	8.43	4.13	2.98
Forecast of RV based on SQFor						Forecast of RV based on $AR(1)$					
$\hat{\beta}_{1,0l}$	4.36	3.31	2.13	1.18	0.83	$\hat{\beta}_{1,0l}$	3.53	3.48	2.87	1.96	1.54
$se\left(\hat{\beta}_{1,0l}\right)$	1.37	0.62	0.47	0.45	0.44	$se\left(\hat{\beta}_{1,0l}\right)$	1.68	1.03	0.66	0.63	0.61
R_l^2	3.39	6.07	4.49	1.99	1.24	R_l^2	1.89	6.33	7.98	5.29	4.03
Subsample: January 1990 to October 2007											
Forecast of RV based on RVFor						Forecast of RV is simply lag RV					
$\hat{\beta}_{1,0l}$	3.32	2.90	1.46	0.36	0.24	$\hat{\beta}_{1,0l}$	4.26	4.70	3.00	1.46	1.08
$se\left(\hat{\beta}_{1,0l}\right)$	1.56	1.15	1.07	1.10	1.08	$se\left(\hat{\beta}_{1,0l}\right)$	2.02	1.01	1.02	1.15	1.02
R_l^2	1.63	4.18	2.14	0.15	0.05	R_l^2	1.15	7.49	5.58	1.24	0.32
Forecast of RV based on SQFor						Forecast of RV based on $AR(1)$					
$\hat{\beta}_{1,0l}$	3.95	3.15	1.59	0.53	0.35	$\hat{\beta}_{1,0l}$	3.76	3.89	2.20	0.69	0.23
$se\left(\hat{\beta}_{1,0l}\right)$	1.64	1.09	1.00	1.09	1.08	$se\left(\hat{\beta}_{1,0l}\right)$	1.80	1.14	1.09	1.23	1.08
R_l^2	2.00	4.27	2.21	0.32	0.13	R_l^2	0.99	5.90	3.34	-0.66	-1.33

Table 3: Long-Run Risk-Return Trade-Offs: January 1930 - December 2012

The entries of the table are the slope coefficient as well as the coefficient of determination (R^2) of the regression

$$\frac{r_{t,t+12h}}{h} = \alpha_{mh} + \beta_{mh}\frac{\sigma^2_{t-12m,t}}{m} + \epsilon^{(m)}_{t,t+h}$$

were $\sigma^2_{t-12m,t}$ is the accumulated past monthly realized variance over the last m years and $r_{t,t+12h}$ is the accumulated future monthly returns over the next h years. Standard errors are corrected for heteroskedasticity and autocorrelation based on the Newey and West (1987) procedure with $12\max(m,h)$ lags.

	$m = h$									
h	1	2	3	4	5	6	7	8	9	10
$\hat{\beta}_{mh}$	0.38	0.55	0.41	-0.16	-0.31	-0.21	0.43	0.74	0.94	1.51
$se\left(\hat{\beta}_{mh}\right)$	0.44	0.32	0.36	0.31	0.43	0.54	0.52	0.51	0.60	0.62
R^2_{mh}	0.30	2.05	1.26	0.02	0.58	0.14	1.33	4.21	6.22	14.49

Table 4: Model Asset Prices Moments

The entries of the table are the first and second moments of consumption and dividend growth rates, the first and second moments of the log price-dividend ratio, the log risk-free rate and excess log equity returns, and finally the slope and R^2 for the regression of 1-year, 3-year and 5-year future excess log equity returns onto the current log price dividend ratio. The first column represents annual data counterparts of these moments over the period from January 1930 to December 2012.

		S1	S2	S3	S4
δ		0.9989	0.9989	0.9989	0.9989
γ		2.5	2.5	2.5	2.5
ψ		1.5	0.75	0.75	0.75
ℓ		2.333	2.333	2.333	2.333
θ		0.989	0.989	0.989	0.989
s_σ/s_{2z}		0.125	0.125	0.125	0.125
$\phi_{1z}^{1/\Delta}$		0.995	0.995	0.999	0.999
k_{2z}		10	10	10	10
$\phi_{2z}^{1/\Delta}$		0.5	0.7	0.5	0.6
ρ		0.40	0.40	0.40	0.40
$\beta_{\rho\sigma}$		0.07	0.07	0.07	0
$E[g_c]$	1.84	1.80	1.80	1.80	1.80
$\sigma[g_c]$	2.20	2.22	2.22	2.22	2.22
$AC1(g_c)$	0.48	0.25	0.25	0.25	0.25
$E[g_d]$	1.05	1.80	1.80	1.80	1.80
$\sigma[g_d]$	13.02	14.26	14.26	14.26	14.26
$AC1(g_d)$	0.11	0.25	0.25	0.25	0.25
$Corr(g_c, g_d)$	0.52	0.40	0.40	0.40	0.40
$E[pd]$	3.33	2.64	2.44	2.77	2.76
$\sigma[pd]$	0.45	0.27	0.27	0.55	0.53
$AC1(pd)$	0.85	0.96	0.96	0.99	0.99
$AC2(pd)$	0.75	0.90	0.90	0.98	0.98
$E[r_f]$	0.65	0.60	1.95	1.97	2.02
$\sigma[r_f]$	3.79	3.91	5.61	4.42	4.95
$E[r]$	5.35	8.62	8.83	7.07	7.00
$\sigma[r]$	20.17	20.55	21.06	20.66	20.58
$\beta(1Y)$	-0.11	-0.19	-0.21	-0.11	-0.11
$R^2(1Y)$	3.53	6.49	6.97	8.49	7.98
$\beta(3Y)$	-0.09	-0.18	-0.20	-0.11	-0.11
$R^2(3Y)$	16.01	16.43	17.20	21.92	20.69
$\beta(5Y)$	-0.09	-0.17	-0.18	-0.11	-0.11
$R^2(5Y)$	23.75	23.32	24.16	31.98	30.32

Table 5: Model Variance Premium Moments

The entries of the table are the first and second moments of the options implied variance, the realized variance and the variance premium. The first column represents daily data counterparts of these moments over the period from January 1990 to December 2012. In computing the daily variance premium, expected realized variance is a statistical forecast of realized variance using the Heterogeneous Autoregressive model of Realized Variance (HAR-RV). The realized variance is the sum of squared daily ($SQFor$) or the sum of squared 5-minute ($\{RVFor\}$) log returns of the S&P 500 index over a 22-day period and its risk-neutral expectation is measured as the end-of-period VIX-squared de-annualized ($VIX^2/12$). All measures are of monthly basis in percentage-squared.

	$SQFor$ $\{RVFor\}$	S1	S2	S3	S4
δ		0.9989	0.9989	0.9989	0.9989
γ		2.5	2.5	2.5	2.5
ψ		1.5	0.75	0.75	0.75
ℓ		2.333	2.333	2.333	2.333
θ		0.989	0.989	0.989	0.989
s_σ/s_{2z}		0.125	0.125	0.125	0.125
$\phi_{1z}^{1/\Delta}$		0.995	0.995	0.999	0.999
k_{2z}		10	10	10	10
$\phi_{2z}^{1/\Delta}$		0.5	0.7	0.5	0.6
ρ		0.40	0.40	0.40	0.40
$\beta_{\rho\sigma}$		0.07	0.07	0.07	0
$E\left[VIX^2\right]$ (1st)		44.93	44.65	43.02	44.03
$\sigma\left[VIX^2\right]$ (1st)		45.45	59.11	45.22	51.71
$E\left[VIX^2\right]$ (2nd)	39.84	56.95	56.12	53.80	53.14
$\sigma\left[VIX^2\right]$ (2nd)	40.23	69.02	78.94	68.06	71.28
$E[RV]$ (1st)		33.52	33.85	31.97	31.39
$\sigma[RV]$ (1st)		64.03	74.28	63.61	68.11
$E[RV]$ (2nd)	30.11 {19.37}	35.10	35.48	33.23	32.41
$\sigma[RV]$ (2nd)	52.53 {33.73}	67.13	77.80	66.16	70.27
$E[VRP]$ (1st)		11.41	10.80	11.05	12.63
$\sigma[VRP]$ (1st)		10.97	10.42	10.96	12.60
$E[VRP]$ (2nd)	9.73 {20.47}	21.85	20.64	20.57	20.73
$\sigma[VRP]$ (2nd)	20.19 {22.08}	2.13	2.02	2.07	1.21

Table 6: Model Short-Run Risk-Return Trade-Offs

The entries of the table are the slope coefficients as well as the coefficient of determination (R_l^2) of the regression

$$\frac{r_{t,t+l}}{l} = \alpha_{0l} + \beta_{1,0l} vp_t + \epsilon^{(0)}_{t,t+l}$$

where vp_t is the current monthly variance premium, and $r_{t,t+l}$ is the accumulated future monthly returns over l months. The first column represents daily data counterparts of these moments over the period from January 1990 to December 2012. In computing the daily variance premium, expected realized variance is a statistical forecast of realized variance using the Heterogeneous Autoregressive model of Realized Variance (HAR-RV). The realized variance is the sum of squared daily ($SQFor$) or 5-minute ($\{RVFor\}$) log returns of the S&P 500 index over a 22-day period and its risk-neutral expectation is measured as the end-of-period VIX-squared de-annualized ($VIX^2/12$).

	$SQFor$ $\{RVFor\}$	S1	S2	S3	S4
δ		0.9989	0.9989	0.9989	0.9989
γ		2.5	2.5	2.5	2.5
ψ		1.5	0.75	0.75	0.75
ℓ		2.333	2.333	2.333	2.333
θ		0.989	0.989	0.989	0.989
s_σ/s_{2z}		0.125	0.125	0.125	0.125
$\phi_{1z}^{1/\Delta}$		0.995	0.995	0.999	0.999
k_{2z}		10	10	10	10
$\phi_{2z}^{1/\Delta}$		0.5	0.7	0.5	0.6
ρ		0.40	0.40	0.40	0.40
$\beta_{\rho\sigma}$		0.07	0.07	0.07	0
$\beta_{1,01}$	4.36 {2.70}	47.57	33.72	51.19	75.60
R_1^2	3.39 {1.54}	2.94	1.31	3.37	2.55
$\beta_{1,02}$		36.65	28.94	38.38	59.83
R_2^2		3.50	1.92	3.76	3.16
$\beta_{1,03}$	3.31 {1.95}	29.36	25.11	29.84	48.27
R_3^2	6.07 {2.49}	3.37	2.16	3.39	3.06
$\beta_{1,04}$		24.35	22.02	23.97	39.65
R_4^2		3.10	2.20	2.90	2.73
$\beta_{1,05}$		20.80	19.51	19.81	33.12
R_5^2		2.82	2.15	2.46	2.37
$\beta_{1,06}$	2.13 {1.90}	18.20	17.45	16.77	28.08
R_6^2	4.49 {4.28}	2.59	2.06	2.11	2.03
$\beta_{1,07}$		16.24	15.74	14.48	24.14
R_7^2		2.40	1.95	1.82	1.74
$\beta_{1,08}$		14.73	14.32	12.72	21.00
R_8^2		2.25	1.84	1.60	1.49
$\beta_{1,09}$	1.18 {1.30}	13.53	13.12	11.32	18.45
R_9^2	1.99 {2.94}	2.14	1.73	1.42	1.29
$\beta_{1,010}$		12.57	12.11	10.20	16.37
R_{10}^2		2.04	1.63	1.27	1.12
$\beta_{1,011}$		11.77	11.24	9.27	14.63
R_{11}^2		1.97	1.54	1.15	0.98
$\beta_{1,012}$	0.83 {1.05}	11.10	10.50	8.50	13.17
R_{12}^2	1.24 {2.44}	1.91	1.46	1.05	0.86

Table 7: Model Long-Run Risk-Return Trade-Offs

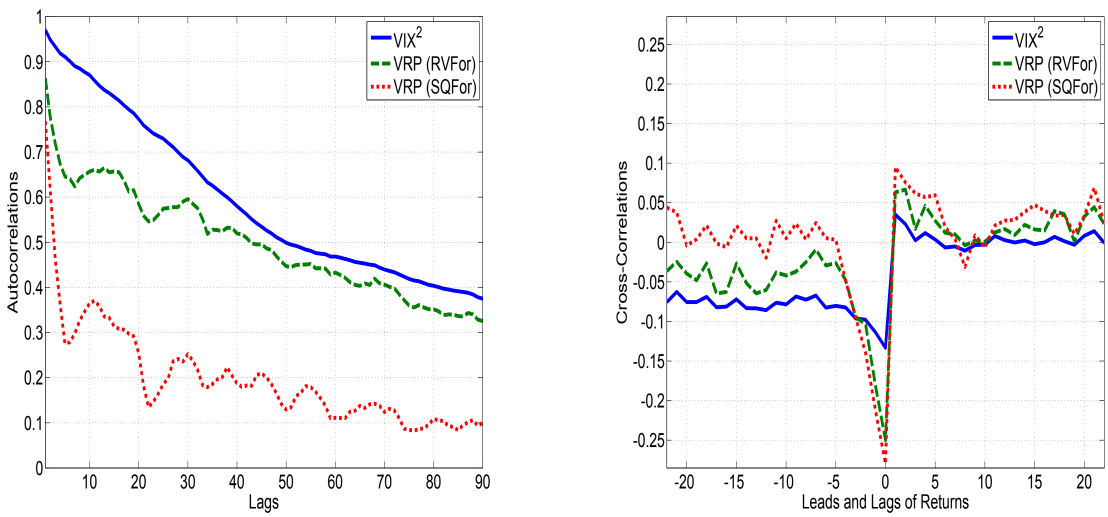
The entries of the table are the slope coefficient as well as the coefficient of determination (R^2) of the regression

$$\frac{r_{t,t+12h}}{h} = \alpha_{hh} + \beta_{hh}\frac{\sigma^2_{t-12h,t}}{h} + \epsilon^{(h)}_{t,t+h}$$

were $\sigma^2_{t-12h,t}$ is the accumulated past monthly realized variance over the last h years and $r_{t,t+12h}$ is the accumulated future monthly returns over the next h years. The first column represents data counterparts of these moments over the period from January 1930 to December 2012, where the monthly realized variance is computed as the sum of squared daily log returns of the S&P 500 index over the month.

		S1	S2	S3	S4
δ		0.9989	0.9989	0.9989	0.9989
γ		2.5	2.5	2.5	2.5
ψ		1.5	0.75	0.75	0.75
ℓ		2.333	2.333	2.333	2.333
θ		0.989	0.989	0.989	0.989
s_σ/s_{2z}		0.125	0.125	0.125	0.125
$\phi_{1z}^{1/\Delta}$		0.995	0.995	0.999	0.999
k_{2z}		10	10	10	10
$\phi_{2z}^{1/\Delta}$		0.5	0.7	0.5	0.6
ρ		0.40	0.40	0.40	0.40
$\beta_{\rho\sigma}$		0.07	0.07	0.07	0
$\hat{\beta}_{11}$	0.38	0.31	0.36	0.17	0.19
R^2_{11}	0.30	1.36	2.14	0.82	0.98
$\hat{\beta}_{22}$	0.55	0.45	0.41	0.28	0.30
R^2_{22}	2.05	3.23	3.37	2.28	2.38
$\hat{\beta}_{33}$	0.41	0.55	0.49	0.38	0.39
R^2_{33}	1.26	5.35	5.05	4.24	4.29
$\hat{\beta}_{44}$	-0.16	0.62	0.54	0.47	0.48
R^2_{44}	0.02	7.35	6.73	6.45	6.45
$\hat{\beta}_{55}$	-0.31	0.67	0.59	0.54	0.55
R^2_{55}	0.58	9.10	8.24	8.75	8.71
$\hat{\beta}_{66}$	-0.21	0.70	0.62	0.61	0.62
R^2_{66}	0.14	10.56	9.52	11.06	10.97
$\hat{\beta}_{77}$	0.43	0.72	0.64	0.67	0.68
R^2_{77}	1.33	11.72	10.56	13.31	13.18
$\hat{\beta}_{88}$	0.74	0.73	0.65	0.72	0.73
R^2_{88}	4.21	12.61	11.37	15.46	15.30
$\hat{\beta}_{99}$	0.94	0.73	0.66	0.76	0.77
R^2_{99}	6.22	13.26	11.97	17.49	17.30
$\hat{\beta}_{10,10}$	1.51	0.73	0.66	0.80	0.81
$R^2_{10,10}$	14.49	13.71	12.40	19.40	19.17

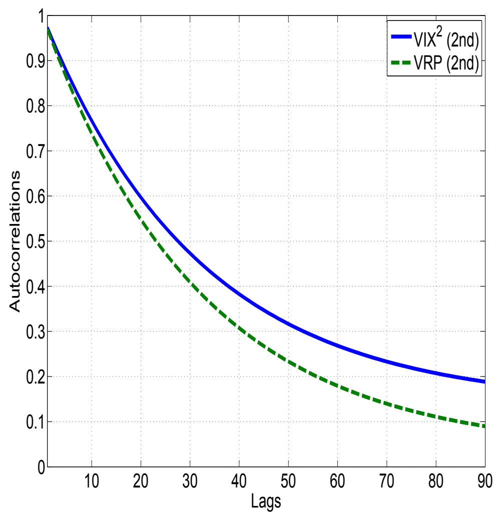
Panel A1 Panel A2



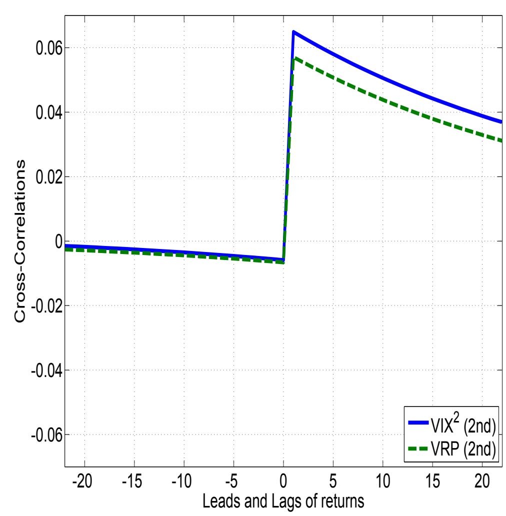
The figure plots the autocorrelations of the daily implied variance and the daily variance premium, as well as their cross-correlations with leads and lags of the daily returns. In computing the daily variance premium, expected realized variance is a statistical forecast of realized variance using the Heterogeneous Autoregressive model of Realized Variance (HAR-RV). The realized variance is the sum of squared 5-minute (RVFor) or the sum of squared daily (SQFor) log returns of the S&P 500 index over a 22-day period and its risk-neutral expectation is measured as the end-of-period VIX-squared de-annualized ($VIX^2/12$). The return series corresponds to excess returns on the S&P 500 index.

Figure 1: **Volatility Effects in Daily Data: January 1990 - December 2012**

Panel A1



Panel A2



The figure plots the model-implied autocorrelations of the daily implied variance, realized variance and variance premium, as well as their cross-correlations with leads and lags of daily returns. The calibration of the consumption and dividends growths dynamics and of the preference parameter values corresponds to the benchmark case.

Figure 2: **Model Volatility Effects**